Lectures on Birational Geometry

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Contents

1 Introduction

Algebraic Geometry is the study of algebraic varieties, their classification. What classification means?

1. attaching numbers and invariants to varieties to distinguish them;
2. grouping varieties according to invariants and characteristics;
3. construct moduli spaces.

Birational Geometry: solving the classification problem using birational technique. The strategy:

1. X projective variety,
2. resolution of singularities $W \to X$,
3. find a nice model $W \overset{\text{bir}}{\dashrightarrow} Y$,
4. classify such Y,
5. get information about original X.

Example 1.1 (Curves)**.** Let X be a projective variety with $\dim X = 1$. Let $W \to X$ be the normalisation = resolution of singularities. Then W uniquely determined. Take $Y = W$. Define $g =$ genus of $Y = h^0(\omega_Y)$. Over the complex setting, it is the number of holes topologically.

All such Y with fixed g form a moduli space M_g. Note if $Y \subseteq \mathbb{P}^2$ is of degree d, then

$$g = \frac{1}{2}(d-1)(d-2).$$

Definition 1.2 (Canonical divisor/bundle)**.** Let X be a smooth projective variety with $\dim X = d$.

Ω_X sheaf of 1-differential forms.

$\omega_X = \wedge^d \Omega_X$ is the canonical sheaf of X.

$\omega_X \simeq \mathcal{O}_X(K_X)$, K_X is the canonical divisor of X.

$K_X =$ divisor of a rational d-differential form.

The canonical divisor/sheaf is of fundamental importance in algebraic geometry just as differential forms are in differential geometry.

Example 1.3.

1. $K_{\mathbb{P}^d} = -(d+1)H$, $H \subseteq \mathbb{P}^d$ hyperplane.
2. $X \subseteq \mathbb{P}^d$, hypersurface of degree l, $K_X = (l-d-1)H|_X$.

Definition 1.4 (Kodaira dimension)**.** X smooth projective variety, $d = \dim X$.

$P_m = h^0(X, mK_X)$ the m-th pluri-genus.

$\kappa(X) =$ Kodaira dimension of X is the largest number in $\mathbb{Z} \cup \{\infty\}$ such that

$$\limsup_{m\to\infty} \frac{h^0(X, mK_X)}{m^{\kappa(X)}} > 0.$$

Can show $\kappa(X) \in \{-\infty, 0, 1, \dots, d\}$.

Example 1.5. X smooth projective curve.

$\kappa(X) = -\infty \iff \deg K_X < 0 \iff X = \mathbb{P}^1 \iff g = 0.$

$\kappa(X) = 0 \iff \deg K_X = 0 \iff X \text{ elliptic} \iff g = 1.$

$\kappa(X) = 1 \iff \deg K_X > 0 \iff X \text{ general type} \iff g \geq 2.$

1.1 Special varieties

Let X projective variety with “good” singularities.

1. X is Fano if $-K_X$ is ample ($\kappa(X) = -\infty$)

2. X is Calabi-Yau if $K_X \equiv 0$ ($\kappa(X) = 0$)

3. X is canonically polarised if K_X is ample ($\kappa(X) = \dim X$)

Exercise 1.6. Examine this classes of varieties for curves.

1.2 A fundamental conjecture

Let X be a projective variety. Then exists birational map $X \dashrightarrow Y$ where Y has good singularities and either

1. Y admits a Fano fibration, or

2. Y admits a Calabi-Yau fibration, or

3. Y is canonically polarised.

1.3 How to find Y?

X projective variety.

$W \to X$ resolution of singularities, (conjectural if char. > 0).

Run the *minimal model program* (MMP) on W;

$$W = W_1 \dashrightarrow W_2 \dashrightarrow W_3 \dashrightarrow \cdots \dashrightarrow W_n = Y.$$

Show Y satisfies Section 1.2.

Establishing this procedure consists of an enormous amount of beautiful ideas, results and conjectures of global and local nature.

Note: the *canonical ring* of W is

$$R(W) \coloneqq \bigoplus_{m \geq 0} H^0(W, mK_W).$$

If $\kappa(X) = \dim W$, then $Y = \operatorname{Proj} R(W)$. If $0 \leq \kappa(W) < \dim W$, then we expect a Calabi-Yau fibration $Y \to \operatorname{Proj} R(W)$. If $\kappa(W) = -\infty$, then we expect a Fano fibration $Y \to Z$.

1.4 The MMP for curves

Let W be a smooth projective curve. Running the MMP on W does not change W. That is, $Y = W$.

1.5 MMP for surfaces

Let W be a smooth projective surface.

$E \subseteq W$ is a -1-curve if $E \simeq \mathbb{P}^1$ and $E^2 = \deg E|_E = -1$.

If exists -1-curve $E \subseteq W$, then exists

$$\begin{tikzcd} W = W_1 \arrow[r, "\mathrm{bir}"] & W_2 \\ \cup & \Cup \\ E \arrow[r] & \mathrm{pt}. \end{tikzcd}$$

If exists -1-curve W_2, we can contract it, etc.

We get

$$W = W_1$$

where Y has no -1-curve.

Can show Y satisfies Section 1.2.

1. $\kappa(W) = -\infty \implies Y \simeq \mathbb{P}^2$, or exists a $\mathbb{P}^1$-bundle $Y \to Z$.
2. $\kappa(W) = 0 \implies K_Y \equiv 0$, Y Calabi-Yau.
3. $\kappa(W) = 1 \implies \exists$ elliptic fibration $Y \to T$.
4. $\kappa(W) = 2 \implies K_Y$ "almost ample", can modify Y, such that K_Y ample.

1.6 Singularities

In dimension 1 and 2, can mostly work with smooth varieties.

In dimension ≥ 3, necessary to allow "good" singularities.

This makes the theory more complicated but much more exciting.

Local behaviour of singularities often corresponding to global behaviour of some varieties.

In dimension 2, we have a detailed classification of "good" singularities.

Example 1.7. $C \subseteq \mathbb{P}^2$ defined by $x^2 + y^2 + z^2 = 0$. (coordinates on $\mathbb{P}^2$)

X = cone over C = surface $\subseteq \mathbb{A}^3$ defined by $x^2 + y^2 + z^2 = 0$. (coordinates on $\mathbb{A}^3$)

Blowing up $(0,0,0)$ on $\mathbb{A}^3$ resolves the singularity $(0,0,0)$ on X:

$$\begin{tikzcd} V \arrow[r] & X \\ \cup & \Cup \\ E \arrow[r] & (0,0,0). \end{tikzcd}$$

where E = exceptional curve $\simeq C$.

1.7 Tools

Birational geometry uses many general and special tools of algebraic geometry.

Resolution of singularities is often used.

Cohomology is used often to proceed by induction.

In particular, the Kodaira vanishing theorem is often used:

Theorem 1.8. *X smooth projective variety (char. $= 0$)*

Let A be an ample divisor on X. Then

$$h^i(X, K_X + A) = 0, \quad \forall i > 0.$$

When char > 0, Frobenius is instead used.

1.8 Some history

- dim 1, many people in 19th century, epically Riemann.
- dim 2, many people in the 19th and early 20th centuries, especially Noether, Enrique, Castelnuovo.
- dim 3, some work by Fano Severi, Zariski, ..., in the first half of 20th century, then big advances by Hironaka, Iitaka, Iskovskikh, Ueno, Shokurov, Mori, Kawamata, Reid, Kollár, Miyaoka, ..., in 1970's – 1990's.
- dim ≥ 4, big advances from 2000's by Shokurov, Hacon-McKernan, Birkar, Cascini, Xu, ...

1.9 Topics in this course

Through the year

1. curves
2. surface
3. singularities
4. pairs
5. explicit geometry, e.g., toric geometry
6. MMP
7. base point freeness, cone and contraction theorems
8. Mori's bend and break technique
9. existence of minimal models
10. finite generation of canonical rings
11. special varieties: Fano, Calabi-Yau and canonically polarised varieties
12. boundedness and moduli of varieties
13. Sarkisov program
14. irrationality of some Fano varieties
15. generalised pairs
16. etc.

1.10 General references

1. Hartshorne, Algebraic geometry
2. Kollár–Mori, Birational geometry of algebraic varieties
3. Birkar, Lectures on birational geometry
4. Matsuki, Introduction to Mori's program
5. Debarre, Higher dimensional algebraic geometry
6. Kawamata-Matsuda–Matsuki, Introduction to the minimal model problem
7. etc.

2 Curves

2.1 Assumption

Let k be an algebraically closed field with characteristics 0. From now on we work over k in this course. Varieties are quasi-projective.

2.2 Recall

Suppose that X is a normal variety. Let $D = \sum d_i D_i$ be a divisor on X. Then $\mathcal{O}_X(D)$ is the sheaf associated to D. For $U \subseteq X$ open subset,

$$\mathcal{O}_X(D)(U) = \{f \in k(X) \mid \operatorname{Div}(f) + D|_U \geq 0\} \cup \{0\}$$

where $K(X)$ is the function field of X, i.e., the field of rational functions on X. Put

$$H^i(X, D) \coloneqq H^i(X, \mathcal{O}_X(D)), \quad h^i(X, D) \coloneqq \dim H^i(X, \mathcal{O}_X(D)).$$

Definition 2.1 (Genus)**.** Let X be a smooth projective curve. The *genus* of X is $g(X) = h^0(X, K_X)$, where K_X is the canonical divisor of X, i.e., the divisor of a rational differential form. We have $\mathcal{O}_X(K_X) = \omega_X = \Omega_X$, the sheaf of differential forms. When $k = \mathbb{C}$, $g(X) =$ the number of holes when we consider X as a Riemann surface.

Example 2.2 ($\mathbb{P}^1$)**.** We have $K_{\mathbb{P}^1} = -2q$, where q is any point on $\mathbb{P}^1$. Since $\deg K_{\mathbb{P}^1} < 0$, there does not exist $f \in k(\mathbb{P}^1)$ such that $\operatorname{Div}(f) + K_{\mathbb{P}^1} \geq 0$. Hence $g(\mathbb{P}^1) = h^0(\mathbb{P}^1, K_{\mathbb{P}^1}) = 0$. When $k = \mathbb{C}$, X is a Riemann sphere (with complex structure).

Theorem 2.3 (Riemann-Roch)**.** *Suppose that X is a smooth projective curve, and $D = \sum d_i D_i$ be a divisor on X. Then*

$$h^0(X, D) - h^0(X, K_X - D) = h^0(X, D) - h^1(X, D) = \deg D + 1 - g(X).$$

Here $\deg D = \sum d_i$.

Proof of Theorem 2.3. First equality follows from Serre duality. For second equality, note that it holds for $D = 0$. Then argue that it holds for D if and only if it holds for $D - x$ for any point $x \in X$, using the exact sequence

$$0 \longrightarrow \mathcal{O}_X(D - x) \longrightarrow \mathcal{O}_X(D) \longrightarrow k \longrightarrow 0$$

and the long exact sequence associated to this. □

Corollary 2.4 (degree of K_X)**.** *Let X be a smooth projective curve. Then* $\deg K_X = 2g(X) - 2$.

Proof. This can be seen by applying Riemann-Roch:

$$g(X) - 1 = h^0(X, K_X) - h^0(X, K_X - K_X) = \deg K_X + 1 - g(X).$$

□

2.3 Types of curves

Let X be a smooth projective curve. Let $\kappa(X)$ be the Kodaira dimension of X. From the Riemann-Roch we can get:

(a) $g(X) = 0 \iff \deg K_X < 0 \iff X \simeq \mathbb{P}^1 \iff \kappa(X) = -\infty$;

(b) $g(X) = 1 \iff \deg K_X = 0 \iff X$ is elliptic $\iff \kappa(X) = 0$;

(c) $g(X) \geq 2 \iff \deg K_X > 0 \iff X$ is general type $\iff \kappa(X) = 1$.

2.4 Elliptic curves

Let X be a smooth projective curve. Let $\mathrm{Div}(X)$ be the group of divisors on X. Let $\mathrm{Cl}(X) = \mathrm{Div}(X)/\sim$ and $\mathrm{Cl}(X)^0 = \{D \in \mathrm{Cl}(X) \mid \deg D = 0\}$.

Now assume $g(X) = 1$, i.e., X is elliptic. Fix a point $y \in X$ and define

$$\alpha\colon X \longrightarrow \mathrm{Cl}(X), \quad x \longmapsto x - y.$$

Then $\alpha(x) = \alpha(x') \implies x - y \sim x' - y \implies x \sim x' \implies \exists f \in k(X)$ such that $\mathrm{Div}(f) = x - x'$. This f must be constant, otherwise f defines a map $f\colon \dashrightarrow \mathbb{P}^1$ of degree 1. So $X \simeq \mathbb{P}^1$, a contradiction. Therefore, $x = x'$.

Also, α is surjective. Take $D \in \mathrm{Cl}(X)^0 \implies \deg(D + y) = 1$, then by Riemann-Roch $h^0(X, D + y) = 1$. There is $f \in k(X)$ such that $\mathrm{Div}(f) + D + y \geq 0$. But $\deg(\mathrm{Div}(f) + D + y) = 1$, so $x \coloneqq \mathrm{Div}(f) + D + y$ is a point. Thus $D \sim x - y = \alpha(x)$.

Therefore, α is bijective. So the group structure on $\mathrm{Cl}(X)^0$ gives a group structure on the set of closed points of X.

Definition 2.5 (Base point free and ample and very ample divisors)**.** Let X be a normal variety, and D a divisor on X. We say that D is *base point free* if $\forall x \in X, \exists f \in k(X)$ such that $x \notin \mathrm{Div}(f) + D \geq 0$ where $f \in H^0(X, D)$. We say that D is *very ample* if there exists an embedding $h\colon X \hookrightarrow \mathbb{P}^n$ such that $D \sim h^*H$ for some hyperplane $H \subseteq \mathbb{P}^n$. We say that D is *ample* if mD is very ample for some $m \in \mathbb{N}$.

Theorem 2.6. *Let X be a smooth projective curve, and D a divisor on X.*

(1) If $\deg D \geq 2g(X)$, then D is base point free.

(2) If $\deg D \geq 2g(X) + 1$, then D is very ample.

(3) If $\deg D > 0$, then D is ample.

Proof. (1) Pick $x \in X$. By Riemann-Roch

$$h^0(X, D) - h^0(X, K_X - D) = \deg D + 1 - g(X),$$
$$h^0(X, D - x) - h^0(X, K_X - D + x) = \deg(D - x) = 1 - g(X) = \deg(D) - g(X).$$

Since $\deg D \geq 2g(X)$,

$$\deg(K_X - D) < 0 \implies h^0(X, K_X - D) = 0,$$
$$\deg(K_X - D + x) < 0 \implies h^0(X, K_X - D + x) = 0.$$

Thus

$$h^0(X, D - x) = h^0(X, D) - 1.$$

This means, there exists some $f \in H^0(X, D)$ such that $x \notin \mathrm{Div}(f) + D$. So D is base point free.

(2) By (1), D is base point free. If $f_0, \dots, f_n$ is a basis of $H^0(X, D)$ as a k-vector space. Then we get a map

$$h\colon X \dashrightarrow \mathbb{P}^n, \quad x \longmapsto (f_0(x) : \cdots : f_n(x)).$$

Since D is free, h is a morphism, and $D \sim h^*H$ for some hyperplane H. Now if $x, y \in X$, similar to (1), we have

$$h^0(X, D - x - y) = h^0(X, D) - 2.$$

In particular, if $x \neq y$, we can find $f \in H^0(X, D)$ such that $x \in \mathrm{Div}(f) + D$ while $y \notin \mathrm{Div}(f) + D$ (essentially says $f(x) = 0$ but $f(y) \neq 0$). This is possible only if h is injective. Moreover, it also implies we can choose the hyperplane H such that h^*H has multiplicity 1 at x. One then shows this is possible inly if $h(X)$ is smooth, so h gives an isomorphism

$$X \longrightarrow h(X) \subseteq \mathbb{P}^1.$$

Hence h is an embedding and D is very ample.

(3) $\deg D > 0 \implies \exists m \in \mathbb{N}$ such that $\deg mD \geq 2g(X) + 1 \implies mD$ is very ample by (2) $\implies D$ is ample. □

Corollary 2.7. *Let X be a smooth projective curve with $g(X) \geq 2$. Then*

(1) K_X is base point free.

(2) $3K_X$ is very ample, so defines an embedding $X \hookrightarrow \mathbb{P}^{5(g-1)-1}$.

Proof. (1) If $x \in X$, then by Riemann-Roch

$$h^0(X, K_X - x) - h^0(X, K_X - K_X + x) = \deg K_X - 1 = 1 - g.$$

But $h^0(X, x) = 1$, otherwise there exists a non-constant $f \in k(X)$ such that $\mathrm{Div}(f) + x \geq 0$. Then $\mathrm{Div}(f) + x$ consists of just one point. So f defines an isomorphism $f\colon X \to \mathbb{P}^2$, a contradiction. Then $h^0(X, K_X - x) = g - 1 = h^0(X, K_X) - 1$. As in the proof of the theorem, this implies K_X is base point free.

(2) $g(X) \geq 2 \implies \deg 3K_X = 3(2g(X) - 2) \geq 2g(X) + 1 \implies 3K_X$ is very ample by the theorem $\implies 3K_X$ defines an embedding $X \hookrightarrow \mathbb{P}^n$ where $n = h^0(X, 3K_X) - 1$. By Riemann-Roch,

$$h^0(X, 3K_X) - h^0(X, K_X - 3K_X) = \deg 3K_X + 1 - g(x) = 5(g(X) - 1).$$ □

Remark 2.8. Using other argument, it is possible to find an embedding $X \hookrightarrow \mathbb{P}^3$ for every smooth projective curve.

2.5 Moduli

Fix $g \geq 0$. Fact: there exists a variety M_g such that

closed points of $M_g \leftrightarrow$ {smooth projective curves of $g(X) = g$, up to isomorphism}.

For example, $M_0 =$ one point. It is well-known that $M_1 = \mathbb{A}^1$. And that $\dim M_2 = 3$. Fact: $\dim M_g = 3g - 3$ for $g \geq 2$.

Example 2.9. Let us take $g = 3$. One shows that for most smooth projective curves of $g(X) = 3$, K_X is very ample, so defines an embedding

$$X \hookrightarrow \mathbb{P}^2.$$

Then one shows $\deg X = 4$ under this embedding. However, this embedding is not unique. Any automorphism of $\mathbb{P}^2$ gives another embedding. The smooth curves in $\mathbb{P}^2$ of degree 4 are parametrised by an open subset $U \subseteq \mathbb{P}^{14}$. This is because the number of coefficients of a general degree 4 homogeneous polynomial is 15. So we have a dominant map $U \to M_3$, where the general fibres have the same dimension as $\mathrm{Aut}(\mathbb{P}^2)$. One then calculates $\dim \mathrm{Aut}(\mathbb{P}^2) = 8$, so $\dim M_3 = 14 - 8 = 6 = 3g - 3$.

3 Surfaces

3.1 Adjunction formula

Let X be a smooth variety, $S \subseteq X$ a smooth prime divisor. Adjunction formula:

$$K_S \sim (K_X + S)|_S.$$

Let $X = \mathbb{P}^n$, and S a hypersurface of degree d. Then $K_X \sim -(n+1)H$ where H is a hyperplane. So, $K_S \sim (K_X + S)|_S \sim (-(n+1)H + dH)|_S = (d-n-1)H|_S$.

If $n = 2$, $\deg K_S = (d-3)d$. Since $\deg K_S = 2g(S) - 2$, we get $g(S) = \frac{1}{2}(d-1)(d-2)$.

3.2 Blowup

Consider $\mathbb{A}^n$ with coordinates $t_1, \ldots, t_n$, and $\mathbb{P}^{n-1}$ with coordinates $a_1, \ldots, a_n$. Let $Y \subseteq \mathbb{A}^n \times \mathbb{P}^{n-1}$ be given by the equations

$$t_i s_j - t_j s_i = 0.$$

Let $\pi \colon Y \to \mathbb{A}^n$ be induced by the projection $\mathbb{A}^n \times \mathbb{P}^{n-1} \to \mathbb{A}^n$. Easy to see that Y is smooth, $\pi^{-1}\{0\} \simeq \mathbb{P}^{n-1}$ (exceptional divisor) and $Y \setminus \pi^{-1}\{0\} \to \mathbb{A}^n \setminus \{0\}$ is an isomorphism.

One can similarly define blowup of $x \in X$ for any smooth variety X using local coordinates. One can also define blowup of a subvariety $V \subseteq X$ for any smooth V, X.

Lemma 3.1. *Let X be a smooth projective surface, $\pi \colon Y \to X$ the blowup of $x \in X$. Let $E := \pi^{-1}\{0\}$ be the exceptional curve. Then*

$$E \simeq \mathbb{P}^1 \quad \text{and} \quad E^2 = -1.$$

Proof. The fact that $E \simeq \mathbb{P}^1$ follows from the definition of blowups. Pick a smooth curve $C \subseteq X$ passing through x. We can see

$$\pi^* C = \widetilde{C} + E$$

and

$$\widetilde{C} \cap E = \text{one point, transversal.}$$

So,

$$(\pi^* C) \cdot E = \deg(\pi^* C)|_E = 0$$
$$(\pi^* C) \cdot E = (\widetilde{C} + E) \cdot E = \widetilde{C} \cdot E + E \cdot E = 1 + E^2.$$

Hence $E^2 = -1$. □

3.3 (-1)-curves

Let X be a smooth surface, $E \subseteq X$ a smooth curve. We say that E is a -1-curve if

$$E \simeq \mathbb{P}^1 \quad \text{and} \quad E^2 = -1.$$

Lemma 3.1 says that exceptional curves of a blowup is a -1-curve.

Theorem 3.2 (Castelnuovo)**.** *Let X be a smooth projective surface, $E \subseteq X$ a curve. Then E is a -1-curve if and only if E is the exceptional curve of a blowup.*

Proof. $\Longleftarrow$ is Lemma 3.1.

$\Longrightarrow$ Idea: find a base point free divisor L. Use its global section to define a morphism $X \to Z$. Ensure that only E is "contracted".

Pick a very ample divisor A such that $H^1(X, A) = 0$. This is possible by Serre vanishing. Put $\ell = A \cdot E$. Claim: $H^1(X, A + iE) = 0, \forall 0 \le i \le \ell$.

Case $i = 0$ holds by assumption.

From the exact sequence

$$0 \longrightarrow \mathcal{O}_X(-E) \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{O}_E \longrightarrow 0$$

we get

$$0 \longrightarrow \mathcal{O}_X(A + (i-1)E) \longrightarrow \mathcal{O}_X(A + iE) \longrightarrow \mathcal{O}_E((A + iE)|_E) \longrightarrow 0. \tag{3.1}$$

The long exact sequence of cohomology gives

$$\cdots \to H^1(X, A + (i-1)E) \to H^1(X, A + iE) \to H^1(E, (A + iE)|_E) \to \dots$$

Note that $\deg(A + iE)|_E = A \cdot E + iE \cdot E = \ell - i \ge 0$ for $0 \le i \le \ell$. So, $H^1(E, (A + iE)|_E) = 0$ for $0 \le i \le \ell$ as $E \simeq \mathbb{P}^1$. Thus

$$H^1(X, A + (i-1)E) = 0 \implies H^1(X, A + iE) = 0, \quad 0 \le i \le \ell.$$

So use induction on i.

Put $L \coloneqq A + \ell E$.

Claim: L is base point free.

Since A is very ample, $\forall x \in X \setminus E$, we can find $s \in H^0(X, L)$ such that $s(x) \ne 0$. So L is base point free outside E. From the exact sequence Eq. (3.1), we get

$$H^0(X, L) \xrightarrow{\lambda} H^0(E, L|_E) \to H^1(X, A + (\ell - 1)E) = 0.$$

So λ is surjective. But $L|_E \sim 0$ is base point free. So for each $x \in E$, we can find $t \in H^0(E, , L|_E)$ such that $t(x) \ne 0$. As λ is surjective, we can find $s \in H^0(X, L)$ such that $s(x) \ne 0$. Thus $L + A + \ell E$ is base point free everywhere.

Now using a basis of $H^0(X, L)$ we can define a morphism

$$\alpha \colon X \to \mathbb{P}^n, \quad n = h^0(X, L) - 1$$

such that $L \sim \alpha^* H$ for some hyperplane $H \subseteq \mathbb{P}^n$. Let Z be the normalisation of $\alpha(X)$. We then get $\pi \colon X \to Z$. Now if $C \subseteq X$ is contracted by π, then $L \cdot C = 0$. But then $C = E$ be definition of L. So, π contracts only E.

Finally one shows that Z is smooth using certain argument, and uses properties of blowups to show that π is the blowup of the point $z \in \pi(E)$. $\square$

3.4 Minimal Model Program (MMP)

Let X be a smooth projective surface. We run the MMP on X. Put $X_1 = X$. If there exists -1-curve $E_1 \subseteq X_1$, we can contract it by Theorem 3.2:

$$X_1 \to X_2.$$

If there exists -1-curve $E_2 \subseteq X_2$, we contract it, and so on

$$X = X_1 \to X_2 \to \cdots \to X_n.$$

Fact: this process stops after finitely many steps. This is essentially for topological reasons. So we get $Y := X_n$ which has no -1-curves. We say that Y is a minimal model of X if K_Y is nef, i.e.,

$$K_Y \cdot C \geq 0, \quad \forall \text{ curves } C \subseteq Y.$$

Later we see that when K_Y is not nef, we have a fibration $Y \to T$ whose general fibres are Fano, $\dim Y > \dim T$. We say that $Y \to T$ is a Mori fibre space for X.

Facts: either $Y \simeq \mathbb{P}^2 \to T = \text{pt.}$, or $Y \to T$ is a $\mathbb{P}^1$-bundle and T is a smooth curve.

Some ideas how to get $Y \to T$. Pick an ample divisor A on Y and let $t \in \mathbb{R}$ be the smallest number such that $K_Y + tA$ is nef. We will see that $t \in \mathbb{Q}^{>0}$ and $m(K_X + tA)$ is base point free for some $m \in \mathbb{N}$.

So, $m(K_Y + tA)$ defines a nontrivial morphism $Y \to T$.

To summarise: running MMP on any smooth projective surface X ends with a smooth projective surface Y,

$$X \dashrightarrow Y$$

such that either Y is a minimal model (K_Y is nef), or there exists a Mori fibre space $Y \to T$ with $Y \simeq \mathbb{P}^2 \to T = \text{pt.}$ or $Y \to T$ is a $\mathbb{P}^1$-bundle with T a smooth curve.

3.5 Classification of surfaces

Let X be a smooth projective surface. Let $X \dashrightarrow Y$ be an MMP. Recall $\kappa(X)$, the Kodaira dimension of X. It is not hard to see that $\kappa(X) = \kappa(Y)$.

If $\kappa(Y) \geq 0$, then Y is a minimal model: $h^0(Y, mK_Y) \neq 0$ for some $m \in \mathbb{N}$ implies that there exists $0 \leq D \sim mK_Y$. Then Y cannot admit a Mori fibre space $Y \to T$. Because $D|_F \sim mK_Y|_F$ for general fibres f of $Y \to T$.

Fact:

$$\kappa(Y) \geq 0 \iff Y \text{ minimal model}$$
$$\kappa(Y) = -\infty \iff Y \text{ admits Mori fibre space } Y \to T$$

Moreover, when $\kappa(Y) \geq 0$, mK_Y is base point free for some $m \in \mathbb{N}$ defining a Calabi-Yau fibration:

$$Y \to V$$

with $\kappa(Y) = \dim V$

One can then attempt to classify the Y for each $\kappa(Y)$. This leads to the moduli theory.

4 Quadric Surfaces

We will study hypersurfaces $X \subseteq \mathbb{P}^3$ of low degree.

4.1 Hyperplanes

Let $X \subseteq \mathbb{P}^3$ be a hypersurface of degree 1. Then X is defined by homogeneous polynomial of degree 1, say F. After linear change of variables, can assume $F = w$, where x, y, z, w are coordinates on $\mathbb{P}^3$. So, $\mathbb{P}^2 \to X \subseteq \mathbb{P}^3$, $(a : b : c) \mapsto (a : b : c : 0)$ is an isomorphism.

4.2 Quadric surfaces

A quadric surface X is a hypersurface $\subseteq \mathbb{P}^3$ of degree 2, that is, defined by an irreducible homogeneous polynomial F of degree 2.

Fact: after linear change of variables, can assume $F = xy - zw$ if X is smooth and $F = x^2 + y^2 + z^2$ if X is singular.

4.3 Singular quadric surface

Let $X \subseteq \mathbb{P}^3$ be defined by $x^2 + y^2 + z^2 = 0$. Then $u = (0 : 0 : 0 : 1)$ is the singular point. In fact, X is the cone over conic $C \subseteq \mathbb{P}^2$ defined by $x^2 + y^2 + z^2 = 0$ ($C \simeq \mathbb{P}^1$).

Let $\pi \colon V \to \mathbb{P}^3$ be the blowup at u, and E be the exceptional divisor ($\simeq \mathbb{P}^2$). Let Y be the birational transform of X. Then $Y \to X$ has one exceptional curve $C \subseteq E \simeq \mathbb{P}^2$, given by $x^2 + y^2 + z^2 = 0$.

$$\begin{tikzcd} C \arrow[r, hook] \arrow[d] & Y \arrow[r, hook] \arrow[d, "\text{bir}"] & V \arrow[d] & E \arrow[l, hook'] \arrow[d] \\ u \arrow[r, hook] & X \arrow[r, hook] & \mathbb{P}^3 & u. \arrow[l, hook'] \end{tikzcd}$$

Note that $C \simeq \mathbb{P}^1$. One can check that Y is smooth. Also,

$$C^2 = (C \cdot C)_Y = ((E|_Y) \cdot C)_Y = (E \cdot C)_V = (E|_E) \cdot C = -L \cdot C = -2,$$

where $L \subseteq E$ is a line. So $Y \to X$ is a resolution of singularities with one exceptional curve C with $C^2 = -2$.

On the other hand, we have that V is a $\mathbb{P}^1$-bundle over $\mathbb{P}^2$.

$$\begin{tikzcd} & V \arrow[dl] \arrow[dr, "\mathbb{P}^1\text{-bundle}"] & \\ \mathbb{P}^3 & & \mathbb{P}^2. \end{tikzcd}$$

From this we get that Y is a $\mathbb{P}^1$-bundle over $\mathbb{P}^1$.

$$\begin{tikzcd} & Y \arrow[dl] \arrow[dr, "\mathbb{P}^1\text{-bundle}"] & \\ X & & \mathbb{P}^1. \end{tikzcd}$$

Exercise 4.1. Show that X is birational to $\mathbb{P}^2$.

4.4 Smooth quadric surface: $\mathbb{P}^1 \times \mathbb{P}^1$

Let $X \subseteq \mathbb{P}^3$ be defined by $xy - zw = 0$. Note hard to show $X \simeq \mathbb{P}^1 \times \mathbb{P}^1$. This is done by considering the Segre map $\alpha\colon \mathbb{P}^1 \times \mathbb{P}^1 - X \subseteq \mathbb{P}^3$, $(a:b),(c:d) \mapsto (ac:bd:bc:ad)$.

4.5 Lines on a smooth quadric

Let $X \subseteq \mathbb{P}^3$ be defined by $xy - zw = 0$. The isomorphism $\mathbb{P}^1 \times \mathbb{P}^1 \to X$ induces two fibrations. For each fibre of either fibration, $\alpha(L) \subseteq \mathbb{P}^3$ is a line., where α is the Segre map above. This means: $\alpha(L) \cdot H = 1$ for a hyperplane $H \subseteq \mathbb{P}^3$. Equivalently, $\alpha(L)$ is the intersection of two hyperplanes in $\mathbb{P}^3$. For example, if $L = \mathbb{P}^1 \times (c:d)$, with c, d fixed. Then

$$\alpha(L) = \{(ac:bd:bc:ad) \mid (a:b) \in \mathbb{P}^1\}$$

is the intersection of the two planes defined by $dx - cw = 0$ and $cy - dz = 0$.

4.6 Class group of a smooth quadric

Let $X \subseteq \mathbb{P}^3$ be defined by $xy - zw = 0$. Then $X \simeq \mathbb{P}^1 \times \mathbb{P}^1$. Let $\mathrm{Cl}(X) =$ groups of divisors on X modulo linear equivalence. We show $\mathrm{Cl}(X) \simeq \mathbb{Z} \times \mathbb{Z}$. Pick any divisor D on $\mathbb{P}^1 \times \mathbb{P}^1$. Let L_1, L_2 be fibres of the two projections. We show that $D\ m_1L_1 + m_2L_2$ for some $m_1, m_2 \in \mathbb{Z}$. Since $L_1 \cdot L_2 = 1$, there exists $m_1 \in \mathbb{Z}$ such that $(D - m_1L_1) \cdot L_2 = 0$. In fact, $(D - m_1L_1) \cdot L = 0$ for any fibre L of the second projection. In particular, $\deg(D - m_1L_1)|_G = 0$ where G is the generic fibre of f_2. So, $D - m_1L_1\ 0$. So, $D - m_1L` \sim B$ for some B such that $B \cap G = \emptyset$. So $B = \sum n_iR_i\ n_i \in \mathbb{Z}$ and R_i are fibres of f_2. But $R_i \sim L_2$ for any i, so $B \sim \sum n_iL_2$. So $D \sim m_1L_1 + m_2L_2$ with $m_2 = \sum n_i$. We have shown that L_1, L_2 generate $\mathrm{Cl}(\mathbb{P}^1 \times \mathbb{P}^1)$. Note that $m_1 = D \cdot L_2$ and $m_2 = D \cdot L_1$. If $m_1L_1 + m_2L_2\ 0$, the $m_1 = m_2 = 0$ by calculating intersection numbers with fibres of f_1, f_2.

Therefore, $Cl(X) \simeq \mathrm{Cl}(\mathbb{P}^1 \times \mathbb{P}^1)$. Hence $X \not\simeq \mathbb{P}^2$.

4.7 Curves on smooth quadric surfaces

Recall: $C \subseteq \mathbb{P}^2$ smooth curve of degree d. Then $g(C) = \frac{1}{2}(d-1)(d-2)$. So there is no smooth curve in P^2 of genus $2, 4, 5, \ldots$.

Now assume $X \subseteq \mathbb{P}^3$ be a smooth quadric surface. Let $C \subseteq X$ be a smooth curve. Then $C\ m_1L_1 + m_2L_2$ for some $m_1, m_2 \in \mathbb{Z}$. We say that C is of bi-degree (m_1, m_2).

On the other hand, $K_{\mathbb{P}^1 \times \mathbb{P}^1} = -2L_1 - 2L_2$ because

$$(K_{\mathbb{P}^1 \times \mathbb{P}^1})|_{L_1} \sim K_{L_1} \quad \text{and} \quad (K_{\mathbb{P}^1 \times \mathbb{P}^1})|_{L_2} \sim K_{L_2},$$

and $(K_{\mathbb{P}^1 \times \mathbb{P}^1}) \cdot L_1 = -2 = (K_{\mathbb{P}^1 \times \mathbb{P}^1}) \cdot L_2$. Thus using adjunction

$$K_C \sim (K_{\mathbb{P}^1 \times \mathbb{P}^1} + C)|_C,$$

we get

$$2g(C) - 2 = \deg K_C = (-2L_1 - 2L_2 + m_1L_1 + m_2L_2) \cdot (m_1L_1 + m_2L_2) = 2m_1m_2 - 2m_1 - 2m_2$$

since $L_1^2 = 0 = L_2^2$ and $L_1 \cdot L_2 = 1$. Therefore, $g(C) = (m_1 - 1)(m_2 - 1)$.

In particular, for any integer $g \geq 0$, we can find a smooth curve $C \subseteq X$ with $g(C) = g$. Choose $m_1, m_2 \in \mathbb{Z}$ such that $g = (m_1 - 1)(m_2 - 1)$, and then choose $C \sim m_1 L_1 + m_2 L_2$. Hence we used

Exercise 4.2. For any $m_1, m_2 \geq 0$, $m_1 L_1 + m_2 L_2$ is base point free.

We also used Bertini's theorem.

Remark 4.3. It is well-known that any smooth projective curve can be embedded in $\mathbb{P}^3$.

Show that every smooth projective curve cannot be embedded in some smooth quadric quadric surface $X \subseteq \mathbb{P}^3$. Use hyperelliptic curves.

4.8 Smooth quadric surfaces are rational

Let $X \subseteq \mathbb{P}^3$ be defined by $xy - zw = 0$. We show that X is birational to $\mathbb{P}^2$, so X is rational. Enough to show that $\mathbb{P}^1 \times \mathbb{P}^1$ is birational to $\mathbb{P}^2$. Well, $\mathbb{A}^1 \times \mathbb{A}^1$ is obvious an open subset of $\mathbb{P}^1 \times \mathbb{P}^1$. But $\mathbb{A}^2 = \mathbb{A}^1 \times \mathbb{A}^1$ is also an open subset of $\mathbb{P}^2$. So we get a birational map $\mathbb{P}^1 \times \mathbb{P}^1 \dashrightarrow \mathbb{P}^2$.

We want to understand this map more detail.

Let L_1, F_1 be the fibres of f_1, and L_2, F_2 the fibres of f_2. Let $u = F_1 \cap F_2$. Note that $F_1^2 = 0 = F_2^2$. Let $\pi\colon V \to \mathbb{P}^1 \times \mathbb{P}^1$ be the blowup on u, and E the exceptional curve. Then $E^2 = -1$, $F_1'^2 = -1$ and $F_2'^2 = -1$. To see this, use $\pi^* F_i = F_i' + E$ and $\pi^* F_i \cdot E = 0$. So, E, F_1', F_2' are (-1)-curves on V. By Theorem 3.2, we can run an MMP on V. First contract $F_1'\colon V \to W$ and then contract $F_2'\colon W \to Y$. Since $\mathrm{Cl}(\mathbb{P}^1 \times \mathbb{P}^1)$ is generate by L_1, L_2. We can check that $\mathrm{Cl}(Y)$ is generated by the birational transform of any of E, L_1, L_2. So, $\mathrm{Cl}(Y) \simeq \mathbb{Z}$. But then again, $Y \simeq \mathbb{P}^2$.

$$\begin{tikzcd}
 & V \arrow[ddl, "\pi"'] \arrow[dr] \arrow[ddrr, dashed, bend left, "\mathrm{MMP}"] & & \\
 & & W \arrow[dr] & \\
X \simeq \mathbb{P}^1 \times \mathbb{P}^1 \arrow[rrr, dashed] & & & Y \simeq \mathbb{P}^2
\end{tikzcd}$$

Smooth quadric surface $X \simeq \mathbb{P}^1 \times \mathbb{P}^1$ is a Fano variety.

5 Cubic Surfaces

We work over an algebraically closed filed k of characteristic 0.

5.1 Cubic surfaces

A *cubic surface* $S \subset \mathbb{P}^3$ is given by a nonzero cubic equation

$$\sum_{i+j+k+l=3,\, i,j,k\geq 0} a_{ijkl}x^i y^j z^k w^l = 0,$$

where $a_{ijkl} \in k$, not all zero.

We usually consider smooth cubic surfaces.

Example 5.1. Fermat cubic surface. $S\colon x^3 + y^3 + z^3 + w^3 = 0$.

5.2 Lines on surfaces

In this section, to see the power of moduli space, we study lines lying on surfaces.

The hypersurfaces of $\mathbb{P}^n$, defined by equations of degree m, are in one-to-one correspondence with points of a projective space $\mathbb{P}^N$, where $N = \binom{n+m}{m} - 1$.

$$\left\{ S\colon \sum_{i+j+k+l=3,\, i,j,k,l\geq 0} a_{ijkl}x^i y^j z^k w^l = 0 \right\} \leftrightarrow \{(\dots, a_{ijkl}, \dots) \in \mathbb{P}^{19}\}$$

The *Grassmannian variety* $\mathrm{Grass}(r, V)$ parametrises r-dimensional vector subspace $L^r \subseteq V$. So the lines in $\mathbb{P}^3$ can be parametrised by $\mathrm{Grass}(2, V)$, where $\dim V = 4$. The Grassmannian variety is a projective variety in $\mathbb{P}(\wedge^r V)$ defined by Plüker equations. In particular, $\mathrm{Grass}(2, V)$ is the quadric hypersurface $\Pi \subset \mathbb{P}^5$ defined by

$$p_{01}p_{23} - p_{02}p_{13} + p_{03}p_{12} = 0.$$

Let

$$\Gamma_m = \{(\xi, \eta) \in \mathbb{P}^N \times \Pi \mid \text{the line } l \text{ corresponding to}$$
$$\Pi \subset \text{ the hypersurface } X \text{ corresponding to } \xi \in \mathbb{P}^N\}.$$

Lemma 5.2 (incidence relation)**.** *$\Gamma_m \subset \mathbb{P}^N \times \Pi$ is closed.*

Proof. Let $l \subset \mathbb{P}^3 = \mathbb{P}(V)$ be a line, where V is a vector space of dimension 4. Then l corresponding to a 2-dimensional subspace L of V. Let $L = \mathrm{Span}\{x, y\}$, where $x, y \in V$ are two vectors.

Let e_0, e_1, e_2, e_3 be a basis of V. Then

$$x = x_0e_0 + x_1e_1 + x_2e_2 + x_3e_3, \quad y = y_0e_0 + y_1e_1 + y_2e_2 + y_3e_3,$$

here $x_i, y_i \in k$. Then $x \wedge y = \sum_{i<y}(x_iy_j - x_jy_i)e_i \wedge e_j$. So the Plüker coordinates of l is $p_{ij} = x_iy_j - x_jy_i$.

All vectors in L has form $xf(y) - yf(x)$, where f runs through all linear functional of V^*. Suppose

$$f = \alpha_0e_0^* + \alpha_1e_1^* + \alpha_2e_2^* + \alpha_3e_3^*,$$

where $\alpha_i \in k$. Then

$$\begin{aligned} & xf(y) - yf(x) \\ = & x(\alpha_0 y_0 + \alpha_1 y_1 + \alpha_2 y_2 + \alpha_3 y_3) - y(\alpha_0 x_0 + \alpha_1 x_1 + \alpha_2 x_2 + \alpha_3 x_3) \\ = & (x_0(\alpha_0 y_0 + \alpha_1 y_1 + \alpha_2 y_2 + \alpha_3 y_3) - y_0(\alpha_0 x_0 + \alpha_1 x_1 + \alpha_2 x_2 + \alpha_3 x_3) + \dots \\ = & z_0 e_0 + z_1 e_1 + z_2 e_2 + z_3 e_3, \end{aligned}$$

where

$$z_i = \sum_j \alpha_j p_{ij}.$$

Let $X_m \subset \mathbb{P}^3$ be a degree m hypersurface defined by

$$F(x, y, z, w) = \sum_{i+j+k+l=m,\ i,j,k,l \geq 0} \beta_{ijkl} x^i y^j z^k w^l = 0.$$

Then $l \subset X_m$ implies that

$$F(z_0, z_1, z_2, z_3) = 0.$$

Plug the two equations, we get an equation of $\beta_{ijkl}, p_{ij}, \alpha_i$. Since $\alpha_i \in k$, the coefficients of the equation vanish. Therefore, we get a system of bi-homogeneous equations of β_{ijkl} and p_{ij}, which define Γ_m. □

There are two natural projections

$$\begin{tikzcd} \mathbb{P}^N \times \Pi \arrow[r, "\varphi"] \arrow[d, "\psi"'] & \mathbb{P}^N \\ \Pi & \end{tikzcd}$$

Since ψ is a proper morphism and $\Gamma_m \subset \mathbb{P}^N \times \Pi$ is closed, $\psi(\Gamma_m) \subset \mathbb{P}^N$ is closed. If $\psi(\Gamma_m) = \mathbb{P}^N$, then every hypersurface of degree m contains at least one line. What remains is to estimate $\dim \Gamma_m = N + 3 - m$ by dimension count. For $m > 3$, $\dim \Gamma_m < N$. We have obtained the following result.

Theorem 5.3. *For any $m > 3$, there exist surfaces of degree m that do not contain any lines. Moreover, such surfaces correspond to an open subset of $\mathbb{P}^N$.*

Example 5.4. Fermat hypersurface F_m of degree m:

$$x^m + y^m + z^m + w^m = 0.$$

Then the line defined by $x + \xi y = z + \xi w = 0$ is contained in F_m, where ξ is a root of $T^m + 1 = 0$.

It is easy to see that surface of degree $m = 1$ or 2 contains infinity many lines.

Theorem 5.5. *Every cubic surface contains at least one line. There exists an open subset U of the space $\mathbb{P}^{19}$ parametrising all cubic surfaces such that a surface corresponding to a point of U contains only finitely many lines.*

Proof. Notice that $\dim \Gamma_3 = 19$. To prove the theorem, we only need to construct a cubic surface contains only finitely many lines. □

Theorem 5.6 (The Theorem on the Dimension of Fibres)**.** *Let $f\colon X \to Y$ be a morphism between varieties. Suppose that f is surjective, $\dim X = n$, $\dim Y = m$. Then $m \leq n$, and* *(i)* $\dim F \geq n - m$ *for any* $y \in Y$ *and for any component* F *of the fibre* $f^{-1}(y)$*.*

(ii) there exists a nonempty open subset $U \subseteq Y$ *such that* $\dim f^{-1}(y) = n - m$ *for* $y \in U$*.*

5.3 Twenty-seven lines on a cubic surface

Let $X \subset \mathbb{P}^3$ be a nonsingular cubic surface. Then X contains a line L by Theorem 5.5. Through L, pass two distinct planes E_1 and E_2 with equations $\varphi_1 = 0$ and $\varphi_2 = 0$, and consider the rational map $\varphi\colon X \to \mathbb{P}^1$ given by $\varphi(x) = (\varphi_1(x) \colon \varphi_2(x))$. The linear system $\lambda_1\varphi_1 + \lambda_2\varphi_2$ corresponding to this map has L as a fixed component: if E_{λ_1,λ_2} is the section of X by the plane with $\lambda_1\varphi_1 + \lambda_2\varphi_2 = 0$ then $E_{\lambda_1,\lambda_2} = L + F_{\lambda_1,\lambda_2}$, where F_{λ_1,λ_2} is a plane conic. The linear system F_{λ_1,λ_2} defines the same map φ. We can show the map φ is regular, and thus $\varphi\colon X \to \mathbb{P}^1$ is a conic bundle.

By genus formula, any line L on a cubic surface has $L^2 = -1$. Indeed, $(K_X + L)\cdot L = L^2 - \deg L = 2g(L) - 2$ and $\deg L = 1$, so $L^2 = -1$.

If L is the line given by $\xi_0 = \xi_1 = 0$ then the equation of X can be written as

$$A(\xi_0,\xi_1)\xi_2^2 + 2B(\xi_0,\xi_1)\xi_2\xi_3 + C(\xi_0,\xi_1)\xi_3^2 + 2D(\xi_0,\xi_1)\xi_2 + 2E(\xi_0,\xi_1)\xi_3 + F(\xi_0,\xi_1) = 0,$$

where A, B, C, D, E and F are forms if ξ_0, ξ_1 of degree $\deg A = \deg B = \deg C = 1$, $\deg D = \deg E = 2$ and $\deg F = 3$.

The degenerate fibres of $\varphi\colon X \to \mathbb{P}^1$ correspond to zeros of the discriminant, each zero has multiplicity 1, and each degenerate fibre is a pair of distinct lines. Then the number of degenerate fibres equals the degree of the discriminant

$$\Delta = \det \begin{vmatrix} A & C & D \\ C & B & E \\ D & E & F \end{vmatrix}$$

which is 5.

Proposition 5.7. *Every line* L *on a nonsingular projective cubic surface* X *meets exactly* 10 *other lines on* X*, which break up into* 5 *pairs of intersecting lines.*

Consider any line L' intersecting L. Similarly, we can consider the projection away from the line L'. Then L' meets 10 lines, of which only L and one further line meet L. Therefore, there exists a line M not intersecting L.

As a conic bundle $\varphi\colon X \to \mathbb{P}^1$, the group $\mathrm{Cl}(X)$ is generated by $L_1, L_2, L_3, L_4, L_5, F, S$, where S is some section of φ. Actually, we can replace S by M, and obtain the following result.

Proposition 5.8. $\mathrm{Cl}(X)$ *is a free group with* 7 *generators, the classes of the line* $L_1, L_2, L_3, L_4, L_5, M$ *and* F*.*

The intersection numbers of $L_1, L_2, L_3, L_4, L_5, M$ and F are tabulated as follows. We now show how to use $\mathrm{Cl}(X)$ to find all the lines on X. A line C on X satisfies $C^2 = -1$. We know L and a further 10 lines intersecting it. We now try to find the lines disjoint from L. These satisfy $CL = 0$, and therefore $CF = 1$.

Suppose $C \sim \sum_{i=1}^5 x_iL_i + yM + zF$. Then $CF = 1$ implies $y = 1$; $C^2 = -1$, $CL = 0$ give

$$-\sum_{i=1}^{5} x_i^2 + 2z = 0, \quad \sum_{i=1}^{5} x_i + 2z = 0.$$

	L_1	L_2	L_3	L_4	L_5	M	F
L_1	-1	0	0	0	0	0	0
L_2	0	-1	0	0	0	0	0
L_3	0	0	-1	0	0	0	0
L_4	0	0	0	-1	0	0	0
L_5	0	0	0	0	-1	0	0
M	0	0	0	0	0	-1	1
F	0	0	0	0	0	1	0

It follows that $\sum_{i=1}^{5}(x_i^2 + x_i) = 0$, that is, each $x_i = 0$ or -1. Moreover, the equation above implies that the numbers of i for which $x_i = -1$ is even, so that either

(a) all $x_i = 0$;

(b) all $x_i = -1$ except one;

(c) $x_i = x_j = -1$, and the remaining $x_k = 0$.

Of these possibilities, (a) gives the class of line M, (b) and (c) 5 and 10 classes. That is, 16 classes altogether. Each class contains at most one line: for if C and C' are distinct lines then $C \cdot C' = 0$ or 1, whereas $C \cdot C' = C^2 = -1$ if $C \sim C'$. Thus, it remains to exhibit at least one line in each class. In case (a), this is M. In case (b), we can show there are L_i'' in the figure. In case (c), we get a class $D_{ij} = -L_i - L_j + M + F$. We can argue by intersection numbers that D_{ij} are different lines from those in the figure.

Hence, there are 17 (in the figure) +10 lines, which are 27 lines on X.

Theorem 5.9. *A nonsingular cubic surface of $\mathbb{P}^3$ has exactly 27 lines.*

5.4 A cubic surface is isomorphic to $\mathbb{P}^2$ blowup 6 points

A nonsingular cubic surface X is a blow up of $Q \simeq P^1 \times \mathbb{P}^1$ for 5 points. Indeed, for the disjoint two lines L and M, we can construct a morphism

$$\varphi = \varphi_L \times \varphi_M : X \to \mathbb{P}^1$$

where φ_L (resp. φ_M) is the conic bundle obtained as the linear projection away from L (resp. M). It is clear that only 5 lines $L_1', \dots, L_5'$ are contracted by φ. So X is $\mathbb{P}^1 \times \mathbb{P}^1$ blown up in five distinct points.

To show a nonsingular cubic surface is isomorphic to $\mathbb{P}^2$ blowup 6 points, we need some preliminaries for linear systems. Let Y be a smooth projective surface, $|D|$ a complete linear system of curves on Y, $P_1, \dots, P_r$ points of Y. Then we will consider the sub-linear system $\mathfrak{d}$ consisting of divisors $D' \in |D|$ which pass through the points $P_1, \dots, P_r$, and we denote it by $|D - P_1 - \cdots - P_r|$. We say that $P_1, \dots, P_r$ are the *assigned base points* of $\mathfrak{d}$.

Let $\pi : Y' \to Y$ be the morphism obtained by blowing up $P_1, \dots, P_r$, and let $E_1, \dots, E_r$ be the exceptional curves. Then there is a natural one-to-one correspondence between the elements of $\mathfrak{d}$ on Y and the elements of the complete linear system $\mathfrak{d}' = |\pi^* D - E_1 - \cdots - E_r|$ on Y' given by $D \mapsto \pi^* D - E_1 - \cdots - E_r$. The new linear system $\mathfrak{d}'$ on Y' may or may not have base points. We call any base point of $\mathfrak{d}'$, considered as an infinitely near point of Y, an *unassigned base point* of $\mathfrak{d}$.

Theorem 5.10. *Let $\mathfrak{d}$ be the linear system of plane cubic curves with assigned base points $P_1, \dots, P_r$. Assume that $P_1, \dots, P_r$ are in general position, that is, no 3 of the P_i are collinear, and no 6 of them lie on a conic. If $r \leq 6$, then the corresponding linear system $\mathfrak{d}'$ on the surface Y' obtained from $\mathbb{P}^2$ by blowing up $P_1, \dots, P_r$ is very ample.*

Corollary 5.11. *With the same hypotheses, for each $r = 0, 1, \dots, 6$, we obtain an embedding of X' in $\mathbb{P}^{9-r}$ as a surface of degree $9 - r$, whose canonical sheaf $\omega_{Y'} \simeq \mathcal{O}_{Y'}(-1)$. In particular, for $r = 6$, we obtain a nonsingular cubic surface in $\mathbb{P}^3$.*

The above corollary shows that we can obtain a cubic surface by blowing up 6 assigned base points of the linear system $|\mathcal{O}_{\mathbb{P}^2}(3)|$ from $\mathbb{P}^2$. On the other hand, for a cubic surface X, we choose any subset of six mutually skew lines $E'_1, \dots, E'_6$ from 27 lines on X. We can blow down these 6 lines and get a smooth projective surface Z, as $(E'_i)^2 = -1$ and Castelnuovo's contraction theorem. Since X is rational and has Picard number 7, Z is a smooth rational surface with Picard number 1. So $Z \simeq \mathbb{P}^2$. It shows that any cubic surface is isomorphic to $\mathbb{P}^2$ blow up 6 points.

5.5 Effective Cone of curves for quadric and cubic surfaces

The effective cone of curves $\mathrm{NE}(Q)$ for a smooth quadric surface Q is spanned by two lines E and F. That is, $\mathrm{NE}(Q) = \{a[E] + b[F] \mid a, b \in \mathbb{R}_{\geq 0}\}$.

Proposition 5.12. *Let X be a smooth cubic surface. Then the effective cone of curves is*

$$\overline{\mathrm{NE}}(X) = \mathrm{NE}(X) = \left\{ \sum_{i=1}^{27} a_i [l_i] \mid a_i \in \mathbb{R}^{\geq 0} \right\}$$

the closed rational polyhedron spanned by the 27 lines $l_1, \dots, l_{27} \subset X$.

Indeed, for a Fano variety Y, the effective cone of curves $\mathrm{NE}(Y)$ is a rational polyhedron cone by the Cone Theorem.

5.6 Rationality for quadric and cubic surfaces

Quadric and cubic surfaces are both del Pezzo surfaces, and rational surfaces. Indeed, they are Fano surfaces follows from adjuction formula, showing their anti-canonical divisors are ample. A smooth quadric Q is isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$, and thus is birational to its open subset $\mathbb{A}^1 \times \mathbb{A}^1$. It shows that Q is a rational surface. For a nonsingular cubic X, the existence of 27 lines, there are two disjoint lines L and M. By Bézout's theorem, a line passing through a point $x \in L$ and a point $y \in M$ will intersect X in $\deg X = 3$ points or the line is contained in X. Hence, it defines a birational map $\mathbb{P}^1 \times \mathbb{P}^1 \dashrightarrow X$, showing that X is a rational surface. Another way to see this is that X is a blow up of rational surfaces Q or $\mathbb{P}^2$. Thus, X is rational.

A more general result is that all del Pezzo surfaces are rational. Indeed, it can be shown from Kodaira vanishing and Castelnuovo's rationality criterion.

Theorem 5.13 (Castelnuovo's rationality criterion)**.** *A smooth projective surface S is rational if and only if*

$$H^1(S, \mathcal{O}_S) = H^0(S, \mathcal{O}_S(2K_S) = 0.$$

Theorem 5.14 (Kodaira vanishing theorem)**.** *Let X be a smooth projective variety over $\mathbb{C}$, and if L is an ample divisor on X, then $H^i(X, K_X + L) = 0$ for all $i > 0$.*

6 Pairs and Singularities 1

One of the successful idea of birational geometry has been to study pairs (X, B) rather than only varieties. Tracing singularities rather than avoiding then has been another key for success.

Why pairs?

Adjunction: Let X be a smooth variety, $S \subseteq X$ a smooth divisor. Then $K_S \sim (K_X + S)|_S$. To lift information from S to X, it is natural to study (X, S).

Canonical bundle formula: Let $f\colon X \to Z$ be a Calabi-Yau fibre space, e.g., elliptic surface. One can write $K_X \equiv f^*(K_Z + B_Z)$ for some $B_Z \geq 0$. To lift information from Z to X, it is natural to study (Z, B_Z).

Quotient varieties: Let X be a variety, G a finite group acting on X. Let $Y = X/G$ and $\pi\colon X \to Y$ the quotient map. We can write $K_X = \pi^*(K_Y + B_Y)$ for some $B_Y \geq 0$. So again natural to study (Y, B_Y).

Why singularities?

We can do birational geometry in dimension two, mostly avoiding singularities, e.g., running MMP on a smooth surface preserves smoothness.

MMP: But in higher dimension, running MMP often involves singular varieties. It is best to treat the singularities.

Singularities are important: Many problems are reduced to understanding singularities, e.g., termination of MMP.

Singularities are interesting: Often behaviour singularities is very interesting and involves deep result and conjectures, e.g., Kawamata log terminal singularities are in some sense local analogous of Fano varieties.

6.1 Pairs

Definition 6.1. A *pair* (X, B) consists of a normal variety X and $B = \sum b_i B_i$ a divisor, $b_i \in [0, 1]$, such that $K_X + B$ is $\mathbb{Q}$-Cartier, i.e., $m(K_X + B)$ is Cartier for some $m \in \mathbb{N}$.

6.2 Log resolution

Resolution of singularities of (X, B) is a projective birational morphism $\phi\colon W \to X$ such that W is smooth, $\phi^{-1} \operatorname{Supp} B \cup \operatorname{Exc}(\phi)$ is a simple normal crossing divisor. Note that $\operatorname{Exc}(\phi) = \cup C$, where $C \subseteq W$ are curves contracted by ϕ. Log resolutions always exist by Hironaka (recall $\operatorname{char} k = 0$)

6.3 Singularities

Let (X, B) be a pair, $\phi\colon W \to X$ a log resolution. We can write $K_W + B_W = \phi^*(K_X + B)$. We say that (X, B) is

1. terminal if each coefficient of B_W is ≤ 0 (< 0 for exceptional components);
2. canonical if each coefficient of B_W is ≤ 0;
3. Kawamata log terminal of each coefficient of B_W is < 1;
4. log canonical if each coefficient of B_W is ≤ 1;
5. ϵ-log canonical if each coefficient of B_W is $\leq 1 - \epsilon$.

For a prime divisor D on W, define $a(D, X, B) := 1 - \operatorname{coeff}_{B_W} D$, which is the *log discrepancy* of D with respect to X.

Lemma 6.2. *The definition of singularities is independent of the choice of log resolution $\phi\colon W \to X$.*

Proof. We check this for log canonical singularities. Others are similar. Assume that (X, B) is log canonical with respect to $\phi\colon W \to X$. Pick another log resolution $\phi'\colon W' \to X$. There is a sequence $\pi\colon Y \to W'$ of smooth blowups such that π is a morphism.

$$\begin{tikzcd} & Y \arrow[dl, "\pi'"'] \arrow[dr, dashed, "\pi"] & \\ W' \arrow[dr, "\phi'"'] & & W \arrow[dl, "\phi"] \\ & X & \end{tikzcd}$$

Writing $K_W + B_W = \phi^*(K_X + B)$ and $K_{W'} + B_{W'} = \phi'^*(K_X + B)$. We have $\pi^*(K_W + B_W) = \pi'^*(K_{W'} + B_{W'})$. So, it is enough to check the case when $W' = Y$ (π' is an isomorphism). Also, it is enough to check the case when $W' \to W$ is one blowup, say blowup of a smooth $Z \subsetneq W$ of codimension c. Then one can calculate $K_{W'} + (1-c)E = \pi^* K_W$ and $\widetilde{B_W} + (\mu_Z B_W)E = \pi^* B_W$. This gives $K_{W'} + B_{W'} = K_{W'} + \widetilde{B_W} + (1 - c + \mu_Z B_W)E = \pi^*(K_W + B_W)$.

As (X, B) is log canonical for $\phi\colon W \to X$, each coefficient of B_W is at most 1. So, $\mu_Z B \le c$ and hence the coefficient of E in $B_{W'}$ is ≤ 1. Thus, (X, B) is log canonical with respect to $\phi'\colon W' \to X$. $\square$

6.4 Examples

(1) Let (X, B) be a log smooth pair: X is smooth, $\operatorname{Supp} B$ is simple normal crossing. Then (X, B) is

1. terminal if each coefficient of B is 0;
2. canonical if each coefficient of B is 0;
3. Kawamata log terminal if and only if each coefficient of B is < 1;
4. log canonical if and only if each coefficient of B is ≤ 1;
5. ϵ-log canonical if each coefficient of B is $\le 1 - \epsilon$.

(2) Let (X, B) be a pair of dimension one. Then (X, B) is log smooth, so we can apply (1).

(3) Let $X = \mathbb{P}^2$, B be a nodal cubic curve. Then (X, B) is log canonical. The blowup of X at the node $x \in B$ is a log resolution: $\phi\colon W \to X$. Note that $\mu_x B = 2$, so one can see $K_W + B_W = \phi^*(K_X + B) = K_W + \widetilde{B} + E$, where E is the exceptional divisor of ϕ. So, (X, B) is log canonical, not Kawamata log terminal.

On the other hand, the pair (X, B') where B' is a curve with one cuspidal singularity is not log canonical.

(4) Let $X = \mathbb{P}^2$ and $B = \frac{2}{3}B_1 + \frac{2}{3}B_2 + \frac{2}{3}B_3$, where B_i are lines through a point x. Again the blow up at x is a log resolution. One can see $K_W + B_W = \phi^*(K_X + B)$ with $B_W = \widetilde{B} + E$, where E is the exceptional divisor of ϕ. So, (X, B) is log canonical but not Kawamata log terminal.

If $C = \frac{1}{3}B_1 + \frac{1}{3}B_2 + \frac{1}{3}B_3$, then (X, C) is Kawamata log terminal.

(5) Let $X \subseteq \mathbb{A}^3$ be defined by $x^2+y^2+z^2=0$. In Section 4.3 we saw that blowing up $(0,0,0)$ gives a resolution of singularities $\phi\colon W \to X$ with one exceptional divisor E with $E^2=-2$. Write $K_W+B_W=\phi^*K_X$ with B_W+bE. Then $(K_W+bE)\cdot E=0$. By adjunction formula, $(K_W+E)\cdot E=-2=\deg K_E$ since $E\simeq\mathbb{P}^1$. So, $K_W\cdot E=0$ as $E^2=-2$.

Thus, $b=0$ which means $(X,0)$ is canonical but not terminal.

(6) Assume X is a surface with a resolution $\phi\colon W\to X$ with one exceptional divisor E with $E^2=-n$ ($n\in\mathbb{N}$). Equivalently, the normal bundle to E in Y has degree $-a$. Such surfaces exist for every n. Write $K_W+B_W=\phi^*K_X$. Similar to (5), we can calculate $B_W=(1-\frac{2}{n})E$. So, $(X,0)$ is Kawamata log terminal. Actually it is $\frac{2}{n}$-log canonical. The larger n, the more singular.

(7) Let $X\subseteq\mathbb{A}^4$ be defined by $xy-zu=0$. This is the cone over the quadric surface in $\mathbb{P}^3$ defined by the same equation. Blowing up $(0,0,0,0)$ gives a resolution $\phi\colon W\to X$ with one exceptional divisor E ($E\subseteq\mathbb{P}^3$ is the quadric surface). One has $E\simeq\mathbb{P}^1\times\mathbb{P}^1$ which comes with two projections.

$$\begin{tikzcd} & E \arrow[dl, "f_1"'] \arrow[dr, "f_2"] & \\ \mathbb{P}^1 & & \mathbb{P}^1 \end{tikzcd} \qquad \begin{tikzcd} E\subseteq W \arrow[r, hook] \arrow[d] & V\supseteq\mathbb{P}^3 \arrow[d, "\text{blow up}"] \\ X \arrow[r, hook] & \mathbb{A}^4\ni(0,0,0,0). \end{tikzcd}$$

Taking a fibre L of f_1 (or of f_2) we can calculate $K_W\cdot L=-1$, $E\cdot L=-1$. So writing K_W+B_W we get $B_W=-E$. Thus, $(X,0)$ has terminal singularities.

$$\begin{tikzcd} & E=\mathbb{P}^1\times\mathbb{P}^1\subseteq W \arrow[dl] \arrow[dr] \arrow[dd] & \\ Y \arrow[rr, dashed] \arrow[dr] & & Y' \arrow[dl] \\ & X & \end{tikzcd}$$

There are two possible contraction of E. One has both Y and Y' are smooth. Also, $Y\to X$ and $Y'\to X$ each contracts one $\mathbb{P}^1$. The birational map $Y\dashrightarrow Y'$ is a *flop*.

6.5 Minimal resolution for surfaces

Let X be a surface, $\phi\colon W\to X$ a resolution. Run an MMP on W over X: contract -1-curves on W which are contracted to points in X. Let Y be the outcome. Then $\psi\colon Y\to X$ is a resolution which is *minimal*, i.e., it does not factor through any other resolution. Here K_Y is nef over X ($K_Y\cdot C\geq 0$ for $C\subseteq Y$ which is contracted to a point on X). Now assume K_X is $\mathbb{Q}$-Cartier and write $K_Y+B_Y=\phi^*K_X$.

Lemma 6.3. *One has $B_Y\geq 0$.*

Proof. This follows from the negativity lemma. □

Corollary 6.4. $(X,0)$ *is terminal if and only if X is smooth.*

Proof. ($\Longleftarrow$) Clear from definition.

($\Longrightarrow$) On the minimal resolution $\psi\colon Y\to X$, $B_Y\geq 0$. But $(X,0)$ is terminal implies each component of B_Y has negative coefficient. This is possible only if ψ is an isomorphism. □

Lemma 6.5. *Let $(X, 0)$ be a Kawamata log terminal pair. Then each exceptional curve of $Y \to X$ (and of $W \to X$) is isomorphic to $\mathbb{P}^1$.*

Proof. Let E be an exceptional curve of $Y \to X$, and b be the coefficient of E in B_Y. Since $(X, 0)$ is Kawamata log terminal, $b < 1$. Since E is exceptional over X, $E^2 < 0$. This can be seen by taking a Cartier divisor $D \geq 0$ on X containing $\psi(E)$ and considering $(\psi^* D) \cdot E = 0$. Note that $(K_Y + B_Y) \cdot E = 0$ and $B_Y \geq 0$. Then $(K_Y + bE) \cdot E \leq 0$. So, $\deg K_E = (K_Y + E) \cdot E \leq (1 - b)E^2 < 0$ by adjunction. Note that here we do not need to know that E is smooth, which still works. This is possible only if $E \simeq \mathbb{P}^1$.

Since $W \to Y$ is a sequence of smooth blowups, each exceptional curve on W is isomorphic to $\mathbb{P}^1$. □

Lemma 6.6 (Negativity lemma)**.** *Let $\psi\colon Y \to X$ be a projective birational morphism of normal varieties. Let D be a $\mathbb{Q}$-Cartier divisor on Y such that*

- *$-D$ is nef over X;*
- *$\psi_* D \geq 0$.*

Then $D \geq 0$.

Proof. First, by localising the problem on X and taking hyperplane sections on Y we can assume that X, Y are surfaces and that f is not an isomorphism. Now after replacing Y with a resolution we can find an effective exceptional divisor E which is anti-nef over X whose support contains $\mathrm{Exc}(f)$. To find such E we could take a nonzero effective Cartier divisor H on X whose support contains the image of $\mathrm{Exc}(f)$. Then, $f^* H = \widetilde{H} + E$ where $\widetilde{H}$ is the birational transform of H and E an exceptional effective divisor. Obviously, $\widetilde{H}$ is nef over X hence E is anti-nef over Y.

Let e be the minimal non-negative number for which $D + eE \geq 0$. If D is not effective, then $D + eE$ has coefficient zero at some exceptional curve C. On the other hand, locally over X, E is connected. So, if $D + eE \neq 0$, we can choose C so that it intersects some component of $D + eE$. But in that case $(D + eE) \cdot C > 0$ which contradicts the assumptions. Therefore, $D + eE = 0$ which is not possible otherwise D and E would be both numerically trivial over X. □

7 Pairs and singularities 2

We continue our study of pairs and singularities. First, let us treat the negativity lemma.

Lemma 7.1 (Negativity lemma)**.** *Let $\psi\colon Y \to X$ be a projective birational morphism of normal varieties. Let D be a $\mathbb{Q}$-Cartier divisor on Y such that*

- *$-D$ is nef over X;*
- *$\psi_* D \geq 0$.*

Then $D \geq 0$.

Proof. Taking a general very ample divisor $H \subseteq X$, letting $G = \psi^* H$, and considering $G \to H$ and $D|_G$, we can reduce the problem to the case $\dim Y = \dim X = 2$. Taking a resolution, can assume Y smooth. Take a Cartier divisor $R \geq 0$ on X containing the image of the exceptional curves. Write $\psi^* R = \widetilde{R} + F$ where $F \geq 0$ and $\operatorname{Supp} F = \cup$exceptional curves. Then $\widetilde{R}$ is nef over X, so $-F$ is nef over X.

Let e be minimal such that $D + eF \geq 0$. If D is not effective, $e > 0$ and there exists a component E of F such that E is not a component of $D + eF$ but E intersects some component of $D + eF$. But then $(D + eF) \cdot E > 0$ contradicting $D \cdot E \leq 0$ and $F \cdot E \leq 0$. So $D \geq 0$. ☐

Definition 7.2 ($\mathbb{Q}$-factorial singularities)**.** Let X be a normal variety. A divisor D on X is $\mathbb{Q}$-Cartier if mD is Cartier for some $m \in \mathbb{N}$. We say that X is $\mathbb{Q}$-*factorial* if every divisor is $\mathbb{Q}$-Cartier. We will see that running MMP preserves $\mathbb{Q}$-factoriality.

Example 7.3. (1) smooth varieties are $\mathbb{Q}$-factorial.

(2) Let X be a surface with Kawamata log terminal singularities. Then X is $\mathbb{Q}$-factorial.

Indeed, assume $\phi\colon W \to X$ is a resolution, and B = sum of exceptional curves. Then (W, B) is log canonical and W is $\mathbb{Q}$-factorial. We will see that we can run an MMP on (W, B) which ends with X, so X is $\mathbb{Q}$-factorial.

(3) Let X be a surface with log canonical singularities. Then X may not be $\mathbb{Q}$-factorial.

Let E be an elliptic curve $\subseteq \mathbb{P}^2$ with cubic equation F. Let X_0 be the cone over E, which is a variety in $\mathbb{A}^3$ defined by F. Define $X \subseteq \mathbb{P}^3$ to be the closure of X_0, which is the projective cone over E. Then X is singular at $(0:0:0:1)$.

Blowing up $(0:0:0:1)$ induces a resolution $\phi\colon W \to X$ with one exceptional curve $C \simeq E$. Fact: there is a $\mathbb{P}^1$-bundle $f\colon W \to E$.

$$\begin{tikzcd} C \arrow[r, hook] \arrow[d, "\text{isomorphism}"'] & W \arrow[dl, "f"] \arrow[dr, "\phi"] & \\ E & & X \end{tikzcd}$$

Now pick a divisor L such that $L \equiv 0$ but $mL \not\sim 0$, $\forall m \in \mathbb{N}$. Let $D = \phi_* f^* L$. Then D is not $\mathbb{Q}$-Cartier: if mD is Cartier, then $\phi^* mD = mf^* L$ (by the Lemma 7.1) and $mf^*L|_C \sim 0$, so $mL \sim 0$, a contradiction. So X is not $\mathbb{Q}$-factorial.

(4) In higher dimension even terminal singularity $\not\Longrightarrow$ $\mathbb{Q}$-factorial.

Let $X \subseteq \mathbb{A}^4$ be defined by $xy - zu = 0$. Recall that X is the cone over a quadric curve. See the diagram below, where ϕ is a resolution with one exceptional divisor $\simeq \mathbb{P}^1 \times \mathbb{P}^1$.

$$
\begin{tikzcd}
 & W \arrow[dl] \arrow[dr] \arrow[dd, "\phi"] & \\
Y \arrow[dr, "f"'] & & Y^+ \arrow[dl, "f^+"] \\
 & X &
\end{tikzcd}
\qquad
\begin{tikzcd}
 & \mathbb{P}^1 \times \mathbb{P}^1 \arrow[dl] \arrow[dr] \arrow[dd] & \\
\mathbb{P}^1 \arrow[dr] & & \mathbb{P}^1 \arrow[dl] \\
 & \mathrm{pt} &
\end{tikzcd}
$$

Pick a prime divisor H on Y interesting the exceptional curve of $Y \to X$ but not containing it. Let $D = f_*H$. Then D is not $\mathbb{Q}$-Cartier: if mD is Cartier, then $(f^*mD) \cdot C = 0$, but $f^*mD = mH$ as f does not contract any divisor, while $mH \cdot C > 0$, a contradiction. So X is not $\mathbb{Q}$-factorial.

7.1 Quotient varieties

Let X be a normal variety. Let $\mathrm{Aut}(X) = \{g \colon X \to X \mid g \text{ isomorphism}\}$ be the automorphism group of X, where the group law is given by composition. Assume that $G \subseteq \mathrm{Aut}(X)$ is finite. Then G acts on X.

Fact: the quotient $Y := X/G$ is a normal variety, and the quotient map $\pi \colon X \to Y$ is a finite morphism.

By the Hurwitz-Riemann formula $K_X = \pi^* K_Y + R$, $R \geq 0$. Now since π is finite, $\pi^* K_Y$ can be defined even if K_Y is not $\mathbb{Q}$-Cartier (Y is smooth in codimension one). Here $R = \sum_{D \text{ prime divisor on } X} (r_D - 1)D$, r_D is the ramification index at D. One can show that R is G-invariant, so there is B_Y such that $K_X = \pi^* K_Y + R = \pi^*(K_Y + B_Y)$.

Exercise 7.4. If X has Kawamata log terminal singularities, e.g., X smooth, then (Y, B_Y) has Kawamata log terminal singularities.

Example 7.5. Let $X = \mathbb{A}^2$, and $\sigma \colon X \to X$, $(a,b) \mapsto (-a,-b)$. Let $G = \langle \sigma \rangle \subseteq \mathrm{Aut}(X)$. Then $|G| = 2$ as $\sigma^2 = \mathrm{id}$. Let $Y = X/G$.

If $X = \operatorname{Spec} k[\alpha,\beta]$, then $Y = \operatorname{Spec} k[\alpha,\beta]^G$ where $k[\alpha,\beta]^G = \{f \in k[\alpha,\beta] \mid f \quad G\text{-invariant}\}$. Note that G acts on $k[\alpha,\beta]$, $\alpha \mapsto -\alpha, \beta \mapsto -\beta$. We can calculate $k[\alpha,\beta]^G = k[\alpha^2, \alpha\beta, \beta^2]$. So $Y = \operatorname{Spec} k[\alpha^2, \alpha\beta, \beta^2]$. Define $\phi \colon k[s,t,u] \to k[\alpha^2, \alpha\beta, \beta^2]$, $s \mapsto \alpha^2, t \mapsto \alpha\beta, u \mapsto \beta^2$. Then $\operatorname{Ker}\phi = \langle su - t^2 \rangle$. So $k[\alpha^2, \alpha\beta, \beta^2] \simeq k[s,t,u]/\langle su - t^2 \rangle$. Then Y is isomorphic to the variety $\subseteq \mathbb{A}^3$ defined by $su - t^2$. So Y is isomorphic to the cone over a conic $\subseteq \mathbb{P}^2$ defined by $su - t^2$.

Exercise 7.6. Calculate B_Y so that $K_X = \pi^*(K_Y + B_Y)$, $\pi \colon X \to Y$ the quotient map.

Remark 7.7. Using quotients we can construct many example of singularities. If X is smooth, singularities on $Y = X/G$ are called *quotient singularities*. In this case Y is $\mathbb{Q}$-factorial.

We can then see that the singularity of $V \subseteq \mathbb{A}^4$ defined by $xy - zu = 0$ is not a quotient singularity because we saw that V is not $\mathbb{Q}$-factorial.

7.2 Vanishing theorems

Theorem 7.8 (Kodaira vanishing)**.** *Let X be a smooth projective variety, A an ample divisor. Then $h^i(X, K_X + A) = 0$, $\forall i > 0$.*

Theorem 7.9 (Kawamata-Viehweg vanishing)**.** *Let (X, B) be a pair with Kawamata log terminal singularities, $f\colon X \to Z$ a projective morphism. Let L be a $\mathbb{Q}$-Cartier divisor on X, $L - (K_X + B)$ is ample (or just nef and big) over Z. Then $R^i f_* \mathcal{O}_X(L) = 0$, $\forall i > 0$.*

In particular, if Z is affine, e.g., $Z = pt$, then $h^i(X, L) = 0$, $\forall i > 0$.

This is one of the most important tools in birational geometry.

7.3 Applying Kawamata-Viehweg vanishing

In birational geometry proofs are often by induction. Here is a rough idea how to apply Theorem 11.4. Let (X, B) be a projective pair with Kawamata log terminal singularities. Let A be an ample divisor. Set $t = \inf\{a \mid K + B + aA \text{ is ample}\}$. Then $K_X + B + tA$ is nef. Assume that $K_X + B$ is not nef, so $t > 0$. Assume $t \in \mathbb{Q}$ (we will see later that this is the case). We will prove that $m(K_X + B + tA)$ is base point free for some $m \in \mathbb{Z}^{>0}$.

A first step is to show $h^0(X, m(K_X + B + tA)) \neq 0$ for some $m \in \mathbb{Z}^{>0}$. The idea is to find Δ, S, L such that

1. (X, Δ) is Kawamata log terminal,
2. L is Cartier,
3. $L - (K_X + \Delta) - S$ is ample,
4. $h^0(X, L) \neq 0 \implies h^0(X, m(K_X + B + tA))$ for some $m \in \mathbb{Z}^{>0}$.

Consider the exact sequence

$$0 \longrightarrow \mathcal{O}_X(L - S) \longrightarrow \mathcal{O}_X(L) \longrightarrow \mathcal{O}_S(L|_S) \longrightarrow 0$$

and the long exact sequence

$$H^0(X, L) \longrightarrow H^0(S, L|_S) \longrightarrow H^1(X, L - S) \longrightarrow 0.$$

By Theorem 11.4, $H^1(X, L - S) = 0$ as $L - S = K_X + B + \text{ample}$. So $H^0(S, L|_S) \neq 0 \implies H^0(X, L) \neq 0$. Now if S is normal and $(X, \Delta + S)$ is log canonical, then $K_S + \Delta_S \coloneqq (K_X + \Delta + S)|_S$ by a general adjunction formula. So $L|_S = K_S + \Delta_S + \text{ample}$. Using induction one makes sure that $H^0(S, L|_S) \neq 0$ and finally derives $h^0(X, m(K_X + B + tA)) \neq 0$ and eventually that $m(K_X + B + tA)$ is base point free.

Why $w$want $m(K_X + B + tA)$ base point free? Because it defines a morphism $X \to Z$ which gives the first step of an MMP on (X, B).

8 Cones of curves and extremal rays

To run MMP on surfaces, need to contract (-1)-curves. It took decade to generalise this to higher dimension. What is the analogue of a (-1)-curve in dimension ≥ 3? Mori answered this introducing extremal rays.

Definition 8.1 ($\mathbb{R}$-1-cycles and $\mathbb{R}$-Cartier divisors)**.** Let X be a projective variety. Let $Z_1(X) =$ group of 1-cycles $= \{\sum n_i C_i \mid n_i \in \mathbb{Z}, C_i \subseteq X \text{ curve}\}$. Define $Z_1(X) \otimes_{\mathbb{Z}} \mathbb{R} =$ group of $\mathbb{R}$-1-cycles, which is an $\mathbb{R}$-vector space. Denote $\operatorname{Pic}(X) =$ group of Cartier divisors modulo linear equivalence. Let $\operatorname{Pic}(X) \otimes_{\mathbb{Z}} \mathbb{R} =$ group of $\mathbb{R}$-Cartier divisors.

For each $D \in \operatorname{Pic}(X) \otimes \mathbb{R}$ and $C \in Z_1(X) \otimes \mathbb{R}$, one can define an intersection number $D \cdot C$. For $D, D' \in \operatorname{Pic}(X) \otimes \mathbb{R}$, $D \equiv D'$ means

$$D \cdot C = D' \cdot C, \quad \forall C \in Z_1(X) \otimes \mathbb{R}.$$

Similarly, for $C, C' \in Z_1(X) \otimes \mathbb{R}$, $C \equiv C'$ means

$$D \cdot C = D \cdot C' \quad \forall D \in \operatorname{Pic}(X) \otimes \mathbb{R}.$$

Define

$$N_1(X) = Z_1(X) \otimes \mathbb{R} / \equiv$$

that is, $\mathbb{R}$-1-cycles modulo numerical equivalence, and

$$N^1(X) = \operatorname{Pic}(X) \otimes \mathbb{R} / \equiv$$

that is, $\mathbb{R}$-Cartier divisors modulo numerical equivalence. Then there is an intersection paring

$$\begin{aligned} N^1(X) \times N_1(X) &\longrightarrow \mathbb{R} \\ (D, C) &\longmapsto D \cdot C \end{aligned}$$

This induces injective $\mathbb{R}$-linear maps

$$\begin{aligned} N^1(X) &\hookrightarrow N_1(X)^* = \text{ dual vector space of } N_1(X), \\ N_1(X) &\hookrightarrow N^1(X)^* = \text{ dual vector space of } N^1(X). \end{aligned}$$

Theorem 8.2 (Theorem of the Base, Néron-Severi)**.** $\dim_{\mathbb{R}} N^1(X) < \infty$.

Therefore, $N^1(X) \simeq N_1(X)^*$, $N_1(X) \simeq N^1(X)^*$. The number $\rho(X) \coloneqq \dim_{\mathbb{R}} N^1(X) = \dim_{\mathbb{R}} N_1(X)$ is the *Picard number* of X.

Definition 8.3 (Cone of curves and extremal rays)**.** Let X be a projective variety. Let $\operatorname{NE}(X) =$ classes in $N_1(X)$ given by $\mathbb{R}$-1-cycles $C \geq 0$. The *Mori-Kleiman* cone of X is

$$\overline{\operatorname{NE}}(X) = \text{closure of } \operatorname{NE}(X) \text{ in } N_1(X).$$

This is a convex cone, i.e.,

- $\alpha \in \overline{\operatorname{NE}}(X), a \in \mathbb{R}^{\geq 0} \implies a\alpha \in \overline{\operatorname{NE}}(X)$.

- $\alpha, \beta \in \overline{\mathrm{NE}}(X) \implies \alpha + \beta \in \overline{\mathrm{NE}}(X)$.

An *extremal face* of $\overline{\mathrm{NE}}(X)$ is a convex subcone F such that $\alpha + \beta \in F \implies \alpha, \beta \in F$ for any $\alpha, \beta \in \overline{\mathrm{NE}}(X)$.

Theorem 8.4 (Kleiman's Ampleness Criterion)**.** *Let D be a $\mathbb{Q}$-Cartier on a projective variety X. Then D is ample if and only if $D \cdot \alpha > 0$ for any $0 \neq \alpha \in \overline{\mathrm{NE}}(X)$.*

Example 8.5. (1) Let X be a projective curve. Then $N^1(X) \simeq \mathbb{R}$, $N_1(X) \simeq \mathbb{R}$ and $\rho(X) = 1$.

(2) Let $X = \mathbb{P}^n$. Then $N^1(X) \simeq \mathbb{R}$, $N_1(X) \simeq \mathbb{R}$ and $\rho(X) = 1$.

(3) Let X be a projective variety, D a nef divisor. Then

$$F_D = \{\alpha \in \overline{\mathrm{NE}}(X) \mid D \cdot \alpha = 0\}$$

is $n$extremal face of $\overline{\mathrm{NE}}(X)$.

Let $f\colon X \to Z$ be a morphism, and A an ample divisor on Z. Then F_{f^*A} is an extremal face of $\overline{\mathrm{NE}}(X)$. A curve $C \subseteq X$ is contracted by f if and only if the class of $C \in F_{f^*A}$. Hence, F_{f^*A} uniquely determines f (up to Stein factorization).

Question 8.6. *Which faces of $\overline{\mathrm{NE}}(X)$ correspond to morphisms $X \to Z$.*

We will see that not every face corresponds to some $X \to Z$. But if X has good singularities, e.g., Kawamata log terminal, then any face F on which K_X is negative corresponds to some $X \to Z$.

(4) Let $\phi\colon X \to \mathbb{P}^n$ be the blowup of a smooth point p_0, and E be the exceptional divisor. Pick a Cartier divisor D on X. Then $\phi_* D \sim mH$ for some $m \in \mathbb{Z}$, where H is a hyperplane. Also, $D = \phi^*\phi_* D + eE$ for some $e \in \mathbb{Z}$. So $D \sim \phi^* mH + eE$. This shows ϕ^*H, E generate $N^1(X)$. So $N^1(X) \simeq \mathbb{R}^2$ and $N_1(X) \simeq \mathbb{R}^2$. Now let $C \subseteq X$ be a curve contracted by ϕ. Then $(\phi^*H) \cdot C = 0$, so the class of C in $\overline{\mathrm{NE}}(X)$ belongs to the extremal face where ϕ^*H vanishes. Another extremal ray is generated by L, the birational transform of a line through p_0.

$$\begin{tikzcd} & X \arrow[dl, "\phi"'] \arrow[dr, "f"', "\mathbb{P}^1\text{-bundle}"] & \\ \mathbb{P}^n & & \mathbb{P}^{n-1} \end{tikzcd}$$

The extremal ray given by C corresponds to ϕ; The extremal ray given by L corresponds to f.

(5) Let Y be a smooth projective variety, and $\mathcal{E}$ a coherent locally free sheaf on Y. Let $f\colon X = \mathbb{P}(\mathcal{E}) \to Y$ be the projective bundle of cE. Then $\mathrm{Pic}(X) \simeq \mathrm{Pic}(Y) \oplus \mathbb{Z}$. So $\rho(X) = \rho(Y) + 1$.

Taking an ample divisor A on Y,the face $F_{f^*A} \subseteq \overline{\mathrm{NE}}(X)$ is an extremal ray of $\overline{\mathrm{NE}}(X)$, generated by curves in the fibres of f.

Now assume Y is a curve. Then $\rho(X) = 2$ and $\overline{\mathrm{NE}}(X)$ has two rays. There are examples where the rays cannot be contracted, i.e., does not correspond to any morphism $X \to Z$.

If $\mathcal{E} = \mathcal{O}_Y \oplus \mathcal{O}_Y(A)$ for some very ample divisor A on Y, then the other ray can be contracted.

$$\begin{tikzcd} & X \arrow[dl] \arrow[dr, "f"] & \\ Y & & Z \text{ cone over } Y. \end{tikzcd}$$

Theorem 8.7. *Let X be a normal projective surface, $C \subseteq X$ a curve that is $\mathbb{Q}$-Cartier. Then,*

- $C^2 < 0 \implies C$ *generates an extremal ray of* $\overline{\mathrm{NE}}(X)$*;*
- $C^2 > 0 \implies C$ *does not belong to any extremal ray, unless* $\rho(X) = 1$*.*

Proof. Assume $C^2 < 0$. Let $\mathcal{C} = \{\alpha \in \overline{\mathrm{NE}}(X) \mid C \cdot \alpha \geq 0\}$. Then $\mathcal{C}$ is a convex cone, and $\overline{\mathrm{NE}}(X)$ is the convex hull of $\mathcal{C}$ and $[C]$ (the class of C). Let $H = \{\alpha \in \overline{\mathrm{NE}}(X) \mid C \cdot \alpha = 0\}$, which is the restriction of a hyperplane. Since $C^2 < 0$, $[C]$ and $\mathcal{C}$ are on different sides of H. Then $[C]$ should be on some extremal ray.

Now assume $C^2 > 0$. Pick $I \in \mathbb{Z}^{>0}$ such that IC is Cartier. Take a resolution of singularities $\phi\colon W \to X$. Then letting $D = \phi^* IC$, we have $D^2 = (IC)^2 > 0$. By the Riemann-Roch theorem,

$$\begin{aligned}\chi(mD) &= h^0(W, mD) - h^1(W, mD) + h^2(W, mD)\\ &= \frac{1}{2}(mD) \cdot (mD - K_W) + \text{constant}.\end{aligned}$$

By Serre duality,

$$h^2(W, mD) = h^0(W, , K_W - mD) = 0 \quad \forall m \gg 0.$$

So $h^0(W, mD)$ grows like m^2. So $h^0(X, mIC) = h^0(W, , mD)$ grows like m^2. Now let A be any curve on X. Consider

$$0 \longrightarrow \mathcal{O}_X(mIC - A) \longrightarrow \mathcal{O}_X(mIC) \longrightarrow \mathcal{O}_A(mIC|_A) \longrightarrow 0$$

which gives an exact sequence

$$0 \longrightarrow H^0(X, mIC - A) \longrightarrow H^0(X, mIC) \longrightarrow H^0(A, mIC|_A) \longrightarrow \dots$$

Since A is a curve, $h^0(A, mIC|_A)$ grows at most like m. So $h^0(X, mIC - A) \neq 0$ for $m \gg 0$. This means there exists some $D \geq 0$ such that

$$mIC - A \sim D \quad \text{for some } m \gg 0.$$

Now if $\rho(X) \neq 1$ and if $[C]$ generates an extremal ray, then $[A]$ belongs to this ray. This is possible only if $\rho(X) = 1$. ∎

Example 8.8. (1) Let X be a smooth quadric surface. Recall that $X \simeq \mathbb{P}^1 \times \mathbb{P}^1$. Considering the projections $f_1, f_2\colon X \to \mathbb{P}^1$, we see $\rho(X) = 2$ and $\overline{\mathrm{NE}}(X)$ has two extremal rays generated by the fibres C_1, C_2 of the two projections f_1, f_2.

(2) Let $X \subseteq \mathbb{P}^3$ be a smooth cubic surface. Then $K_X = (K_{\mathbb{P}^3} + X)|_X \sim -H|_X$, where H is a hyperplane. Recall that X contains 27 lines. If $L \subseteq X$ is any line, then $K_X \cdot L = -H \cdot L = -1$. So from

$$-2 = \deg K_L = (K_X + L) \cdot L = -1 + L^2$$

we see that $L^2 = -1$. So L is a (-1)-curve and it can be contracted. Then $[L] \in \overline{\mathrm{NE}}(X)$ generates an extremal ray.

It is well-known that contracting 6 of the lines carefully we reach $\mathbb{P}^2$. In particular, $\rho(X) = 7$. Now $-K_X$ is ample, so by Theorem 8.4, K_X is negative on every extremal ray. Moreover, we will see that every extremal ray corresponds to some morphism $X \to Z$. Working a bit one can see that the extremal rays of X are exactly those generated by the lines.

9 Linear systems and Kodaira dimension

Definition 9.1 (Divisorial sheaves)**.** Let X be a normal variety. We can interpret sections of divisors on X as rational functions. If D is a Weil divisor on X, then we can describe $\mathcal{O}_X(D)$ in a canonical way as

$$\mathcal{O}_X(D)(U) = \{f \in K(X)^* \mid (f) + D|_U \geq 0\} \cup \{0\}$$

where U is an open subset of X and $K(X)$ is the function field of X. If $D' \sim D$, then of course $\mathcal{O}_X(D') \simeq \mathcal{O}_X(D)$ but these sheaves are not canonically identical, i.e., they have different embedding in the constant sheaf associated to $K(X)$.

If D is Cartier, then $\mathcal{O}_X(D)$ is isomorphic to the sheaf defined in Hartshorne.

If W is any open subset of X such that $\operatorname{codim} X \setminus W \geq 2$, then $\mathcal{O}_X(D) = j_*\mathcal{O}_W(D|_W)$, where $j \colon W \hookrightarrow X$ is the inclusion, because

$$(f) + D \geq 0 \iff (f) + D|_W \geq 0.$$

In particular, by taking $W = X \setminus X_{\text{sing}}$ one can easily see that the above sheaves are reflexive.

Definition 9.2 (Base locus, maps of divisor)**.** Let X be a normal variety, and D a divisor. The *linear system* of D is

$$|D| = \{D' \geq 0 \mid D \sim D'\} = \{\operatorname{Div}(f) + D \mid F \in H^0(X, \mathcal{O}_X(D))\}.$$

The *base locus* is

$$\operatorname{Bs}|D| = \bigcap_{D' \in |D|} \operatorname{Supp} D'$$

If $H^0(X, D) \coloneqq H^0(X, \mathcal{O}_X(D)) = 0$, we let $\operatorname{Bs}|D| = X$. The *fixed part* of $|D|$ is the largest divisor $F \geq 0$ such that $F \leq D'$, $\forall D' \in |D|$. The *movable part* of $|D|$ is $D - F$.

When $0 < \dim_k H^0(X, D) < \infty$, e.g., X is projective, choosing a basis $f_1, \ldots, f_n$ of $H^0(X, D)$, we can define a *rational map*

$$\phi_D \colon X \dashrightarrow \mathbb{P}^{n-1}, \quad x \mapsto (f_1(x) : \cdots : f_n(x)).$$

Note that ϕ_D is defined on $X \setminus \operatorname{Bs}|D|$. When $\operatorname{Bs}|D| = \emptyset$, D is *base point free* and ϕ_D is a morphism.

Definition 9.3 (Contraction)**.** Let $f \colon X \to Z$ be a projective morphism of varieties (or schemes). This f is a *contraction* if $f_*\mathcal{O}_X = \mathcal{O}_Z$. This implies the fibres of f are connected.

Theorem 9.4 (Stein factorisation)**.** *Let $g \colon X \to V$ be a projective is a projective morphism of varieties. Then g factors as*

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[rr, bend right, "g"'] & Z \arrow[r, "h"] & V \end{tikzcd}$$

where f is a contraction and h is finite.

Definition 9.5 (Kodaira dimension)**.** For a divisor D on a normal projective variety X, define the *Kodaira dimension* of D as the largest number

$$\kappa(D) \in \{-\infty\} \cup \mathbb{Z}$$

satisfying

$$0 < \limsup_{m \to +\infty} \frac{h^0(X, mD)}{m^{\kappa(D)}}$$

This says the function $h^0(X, mD)$ of m "grows like $m^{\kappa(D)}$ ".

If D is a $\mathbb{Q}$-divisor, we define

$$\kappa(D) \coloneqq \kappa(\ell D)$$

for some $\ell \in \mathbb{Z}^{>0}$ such that ℓD is integral. By the next lemma, this is well-defined.

Lemma 9.6. *Let X be a normal variety, and D, D' divisors.*

(1) $\kappa(D) = \kappa(nD)$ for any $n \in \mathbb{N}$.

(2) If $nD \sim nD'$ for some $n \in \mathbb{N}$, then $\kappa(D) = \kappa(D')$.

Proof. (1) From the definition, it is clear that $\kappa(D) \geq \kappa(nD)$. If $h^0(X, mD) = 0$ for some $m > 0$, then

$$h^0(X, mnD) \geq h^0(X, mD).$$

If $h^0(X, mD) \neq 0$ for some $m > 0$, then $mD \sim G$ for some $G \geq 0$. We still have

$$h^0(X, mnD) = h^0(X, nG) \geq h^0(X, G) = h^0(X, mD).$$

So, if $h^0(X, mD)$ "grows like $m^{\kappa(D)}$ ", then so does $h^0(X, mnD)$. Hence $\kappa(D) = \kappa(nD)$.

(2) $nD \sim nD' \implies \kappa(nD) = \kappa(nD') \implies \kappa(D) = \kappa(D')$ by (1). $\square$

Lemma 9.7. *Let X be a normal projective variety, and D a divisor. Then*

$$\kappa(D) \in \{\infty, 0, 1, \ldots, \dim X\}.$$

If D is ample, then $\kappa(D) = \dim X$.

Proof. By definition,

$$\kappa(D) \in \{\infty, 0, 1, \ldots\}.$$

So it is enough to show that $\kappa(D) \leq \dim X$. Taking a resolution, we may assume that X is smooth. (Assume D is $\mathbb{Q}$-Cartier?)

First assume D is very ample. Changing D linear, we may assume D is smooth (Bertini theorem). Now consider the exact sequence

$$0 \longrightarrow \mathcal{O}_X((m-1)D) \longrightarrow \mathcal{O}_X(mD) \longrightarrow \mathcal{O}_D(mD|_D) \longrightarrow 0$$

which gives the exact sequence

$$0 \longrightarrow H^0(X, (m-1)D) \longrightarrow H^0(X, mD) \longrightarrow H^0(D, mD|_D) \longrightarrow \ldots$$

By induction, $\kappa(D|_D) = \dim D = \dim X - 1$, so $h^0(D, mD|_D)$ "grows like $m^{\dim X - 1}$ ". Thus $h^0(X, mD)$ "grows like $m^{\dim X}$.

(Note that when D is very ample,

$$\chi(X, mD) = h^0(X, mD), \quad \forall m \gg 0,$$

so $h^0(X, mD)$ is just the Hilbert polynomial of X for large m.)

Now assume D is arbitrary. Assume A is an ample divisor. Then $h^0(X, nA - D) \neq 0$ for $n \gg 0$. So $nA \sim D + G$ for some $G \geq 0$. Then $\kappa(D) \leq \kappa(nA) = \kappa(A) = \dim X$. $\square$

Example 9.8 (Curves)**.** Let X be a smooth projective curve and D a $\mathbb{Q}$-divisor on X.

- $\deg D < 0 \implies h_0(X, mD) = 0,\ \forall m \in \mathbb{Z}^{>0} \implies \kappa(D) = -\infty$.
- $\deg D > 0 \iff D$ ample $\iff \kappa(D) = 1$.
- $\deg D = 0 \implies$
  - $\kappa(D) = 0 \iff D$ is torsion, i.e., $mD \sim 0$ for some $m \in \mathbb{Z}^{>0}$.
  - $\kappa(D) = -\infty \iff D$ is not torsion.

Here is another classification.

- $\kappa(D) = -\infty \iff \deg D < 0$, or $\deg D = 0$ and D is not torsion.
- $\kappa(D) = 0 \iff \deg D = 0$ and D is torsion.
- $\kappa(D) = 1 \iff \deg D > 0 \iff D$ is ample.

Example 9.9 (Exceptional divisor)**.** Let $f\colon X \to Y$ be a birational contraction of normal projective varieties. Let $D \geq 0$ be a divisor on X which is exceptional over Y, i.e., $f_*D = 0$. Then $\kappa(D) = 0$: We have an injection $H^0(X, mD) \hookrightarrow H_0(Y, mf_*D) = H^0(Y, 0) = k$. So $h^0(X, mD) \leq 1$, $\forall m \in \mathbb{Z}^{>0}$. Since $D \geq 0$, $h^0(X, mD) = 1$, $\forall m \in \mathbb{Z}^{>0}$. So $\kappa(D) = 0$.

Definition 9.10 (Big divisor)**.** A $\mathbb{Q}$-divisor D on a normal projective variety X is called *big* if $\kappa(D) = \dim X$. Every ample divisor is big.

Theorem 9.11 (Kodaira lemma)**.** *Let D, L be $\mathbb{Q}$-divisors on a normal projective variety X where D is big and $\mathbb{Q}$-Cartier. Then, there is a rational number $\epsilon > 0$ such that $D - \epsilon L$ is big.*

Proof. We may assume that X is smooth and D is Cartier. Take a general very ample divisor A and consider

$$0 \longrightarrow \mathcal{O}_X(mD - A) \longrightarrow \mathcal{O}_X(mD) \longrightarrow \mathcal{O}_A(mD|_A) \longrightarrow 0$$

to deduce $h^0(X, mD - A) \neq 0$ for some $m \gg 0$. Next, pick $n \gg 0$ such that $nA - L$ is ample. Then

$$\kappa(mnD - L) = \kappa(mnD - nA + nA - L) \geq \kappa(nA - L) = \dim X.$$

So, $mnD - L$ is big, hence $D - \frac{1}{mn}L$ is big. ☐

Corollary 9.12. *Let D be a nef $\mathbb{Q}$-Cartier divisor on a normal projective variety X. Then, the following are equivalent:*

(1) D is big,

(2) there is an effective $\mathbb{Q}$-divisor G and ample $\mathbb{Q}$-divisors A_m such that $D \sim_{\mathbb{Q}} A_m + \frac{1}{m}G$ for every m.

Proof. ($\Longleftarrow$) $\kappa(D) > \kappa(D - \frac{1}{m}G) = \dim X$.

($\Longrightarrow$) Pick an ample divisor A. By Theorem 9.11, there is some $t \in \mathbb{Q}^{>0}$ such that $D - tA$ is big. So, $h^0(X, \ell(D - tA)) \neq 0$ for some $\ell \in \mathbb{Z}^{>0}$. Then $\ell D - \ell tA \sim N \geq 0$, and hence $D - tA \sim_{\mathbb{Q}} G := \frac{1}{\ell}N$. Now, $\forall m \in \mathbb{Z}^{>0}$,

$$D - \frac{1}{m}G = D - \frac{1}{\ell}N = (1 - \frac{1}{m})D + \frac{1}{m}(D - \frac{1}{\ell}N)$$

is ample by Kleiman's ampleness criterion because $(1 - \frac{1}{m})D$ is nef and $\frac{1}{m}(D - \frac{1}{\ell}N)$ is ample. ☐

Definition 9.13 (Kodaira dimension of varieties)**.** Let X be a normal projective variety. The Kodaira dimension of X is $\kappa(X) := \kappa(K_X)$.

In dimension one, genus is the most important invariant. In higher dimension, it is the Kodaira dimension that is important.

Example 9.14. Let X be a smooth projective curve of genus g. Recall $g = h^0(X, K_X)$.

- $\kappa(X) = -\infty \iff g = 0 \iff X \simeq \mathbb{P}^1 \iff \deg K_X < 0$;
- $\kappa(X) = 0 \iff g = 1 \iff X$ elliptic $\iff \deg K_X = 0$.
- $\kappa(X) = 1 \iff g \geq 2 \iff X$ general type $\iff \deg K_X > 0$.

Theorem 9.15 (Iitaka fibration)**.** *Let D be a $\mathbb{Q}$-Cartier divisor on a normal projective variety X with $\kappa(D) \geq 0$. Then, there are projective morphisms $f\colon W \to X$ and $g\colon W \to Z$ from a smooth W,*

$$\begin{tikzcd} & W \arrow[dl, "f"'] \arrow[dr, "g"] & \\ X & & Z \end{tikzcd}$$

such that

- *f is birational,*
- *g is a contraction,*
- *$\kappa(D) = \dim Z$, and*
- *if V is the generic fibre of g, then $\kappa(f^*D|_V) = 0$.*

This is very useful for induction on dimension.

Remark 9.16. Let D be a Cartier divisor on a normal projective variety X. It is well-known that

$$\kappa(D) = \max\{\dim \phi_{mD}(X) \mid h^0(X, mD) \neq 0\}$$

if $h^0(X, mD) \neq 0$ for some $m \in \mathbb{N}$.

Now assume X has Kawamata log terminal singularities. Consider the case $D = K_X$ with $\kappa(K_X) \geq 0$. Conjectures in birational geometry say that we can run an MMP on X giving

$$\begin{tikzcd} X \arrow[r, dashed, "\mathrm{MMP}"] & Y \arrow[d, "g"] \\ & Z \end{tikzcd}$$

where Y is projective and $K_Y \sim_{\mathbb{Q}} g^*A$ for some ample $\mathbb{Q}$-divisor A on Z. Moreover, $\kappa(X) = \kappa(Y) = \kappa(A) = \dim Z$. If G is a general fibre of g, $K_G \equiv 0$ and $\kappa(G) = 0$. Taking a common resolution

$$\begin{tikzcd} & W \arrow[dl] \arrow[dr] \arrow[ddr, bend left] & \\ X & & Y \arrow[d, "g"'] \\ & & Z \end{tikzcd}$$

we get the Iitaka fibration. Now assume $\dim X = 2$.

- $\kappa(X) = 0 \iff \dim Z = 0 \iff K_Y \equiv 0 \iff Y$ Calabi-Yau.
- $\kappa(X) = 1 \iff \dim Z = 1 \iff Y \to Z$ elliptic fibration and $K_Y \sim_{\mathbb{Q}} g^*(\text{ample})$.
- $\kappa(X) = 2 \iff \dim Z = 2 \iff K_Y$ big.

When $\kappa(X) = -\infty$, the MMP produces a Mori-Fano fibration.

10 The LMMP and some open problems

Definition 10.1 (Contraction of an extremal ray)**.** Let X be a normal projective variety, $\overline{\mathrm{NE}}(X)$ the Kleiman-Mori cone of X. Let F be an extremal face of $\overline{\mathrm{NE}}(X)$, e.g., an extremal ray. We say that F is *contractible* if there exists a contra $f\colon X \to Z$ and an ample divisor A on Z such that

$$F = F_{f^*A} \coloneqq \{\alpha \in \overline{\mathrm{NE}}(X) \mid f^*A \cdot \alpha = 0\}.$$

In particular,

$$f(C) = \text{pt.} \iff [C] \in F, \quad \forall \text{ curve } C \subseteq X.$$

for any curve $C \subset X$. We say that f is the contraction of F.

10.2 (Types of contractions)**.** Assume R is an extremal ray, contracted by $f\colon X \to Z$. We have these types:

Divisorial: f is birational and it contracts at least some divisors.

Small: f is birational and it does not contracts divisors.

Fibration: f is not birational hence $\dim X > \dim Z$.

Definition 10.3 (Log minimal model program)**.** Let (X, B) be a projective pair. If $K_X + B$ is not nef, then there is an extremal ray R such that

$$K_X + B \cdot R < 0.$$

Assume that R can be contracted, say by $f\colon X \to Z$. If $X \to Z$ is a fibration, we stop. Assume not. If $X \to Z$ is a divisorial contraction, let $B_Z = f_*B$, and continue with (Z, B_Z) as before. If $X \to Z$ is small, we say that this is a flipping contraction. Assume the flip of $X \to Z$ exists, i.e., there exists

$$\begin{tikzcd} X \arrow[rr, dashed, "\lambda"] \arrow[dr] & & X^+ \arrow[dl] \\ & Z & \end{tikzcd}$$

where $X^+ \to Z$ is small, and $K_{X^+} + B^+ = \lambda_*(K_X + B)$ is ample over Z. That is, $K_X + B$ is negative over Z, and $K_{X^+} + B^+$ is positive over Z. (Note that in the flip case, $K_Z + B_Z$ is never $\mathbb{Q}$-Cartier.) If the flip exists, continue with (X^+, B^+) as before. This process is the *log minimal model program* (LMMP) on (X, B).

If the program stops, we get a model (Y, B_Y)

$$(X, B) \dashrightarrow \ \dashrightarrow \cdots \dashrightarrow \ \dashrightarrow (Y, B_Y)$$

such that either

- $K_Y + B_Y$ is nef: (Y, B_Y) is a *minimal model*; or
- there is a fibre type contraction $Y \to Z$: *Mori fibre space* (or Fano fibration). (In this case, $K_Y + B_Y$ is negative over Z.)

Example 10.4. Let X be a smooth projective surface, and $B = 0$. The LMMP on X is just the minimal model described in Section 3. The program ends with Y such that either

- Y is a minimal model; or
- there is a Mori fibre space $Y \to Z$:

– $Y = \mathbb{P}^2 \to Z = \text{pt.}$,

– $Y \to Z$ is a $\mathbb{P}^1$-bundle over a smooth curve.

Remark 10.5. Assume (X, B) is projective with log canonical singularities. Then

- Every step of the LMMP exists.
- The LMMP is not unique.

Conjecture 10.6 (Termination)**.** *Let (X, B) be a projective log canonical pair. Then every LMMP on (X, B) terminates with a minimal model or a Mori fibre space.*

Remark 10.7. Known up to dimension 3 [Kawamata, Mori, Reid, Shokurov, etc].

Known in any dimension if (X, B) is Kawamata log terminal and $K_X + B$ is big [BCHM]. (Some choice of the LMMP terminates.)

Conjecture 10.8 (Abundance)**.** *Let (X, B) be a projective log canonical pair, and $K_X + B$ nef. Then $K_X + B$ is semi-ample, i.e., there is a contraction $g\colon X \to T$ such that $K_X + B \sim_{\mathbb{Q}} g^*(\text{ample})$. ($\iff$ $m(K_X + B)$ is free for some $m \in \mathbb{N}$.)*

Remark 10.9. Known up to dimension 3 [Miyaoka, Kawamata, Keel-Matsuki-McKernan].

Known in any dimension when (X, B) is Kawamata log terminal, $K_X + B$ is big [Kawamata, Shokurov].

A closely related conjecture is the following:

Conjecture 10.10 (Nonvanishing)**.** *Let (X, B) be a projective log canonical pair, and $K_X + B$ pseudo-effective, i.e.,*

$$K_X + B + tA \text{ is big} \quad \forall t > 0, \forall \text{ ample divisor } A.$$

Then the Kodaira dimension

$$\kappa(K_X + B) \geq 0.$$

Remark 10.11. Known up to dimension 3 [Miyaoka].

Conjecture 10.12 (Finite generation)**.** *Let (X, B) be a projective log canonical pair. Then*

$$R(X, B) \coloneqq \oplus_{m \geq 0} H^0(X, \lfloor m(K_X + B) \rfloor)$$

is a finitely generated k-algebra (k = ground field).

Remark 10.13. Known up to dimension 4.

Known in any dimension when (X, B) is Kawamata log terminal [BCHM].

Conjecture 10.14 (Iitaka)**.** *Let $f\colon X \to Z$ be a contraction of smooth projective varieties. Then*

$$\kappa(K_X) \geq \kappa(K_F) + \kappa(K_Z)$$

where F is a general fibre of f.

Example 10.15 (Divisorial contractions)**.** (1) Let Z be a smooth projective variety, $f\colon X \to Z$ a blowup of a smooth subvariety $V \subseteq Z$, and E the exceptional divisor. One can calculate

$$K_X = f^*K_Z + (c-1)E, \quad c = \operatorname{codim}_Z V.$$

Here f is the contraction of the extremal ray R on X generated by $[C]$, $C \subseteq E$, $f(C) = \text{pt}$.

(2) Let $\mathcal{E} = \mathcal{O}_{\mathbb{P}^n} \oplus \mathcal{O}_{\mathbb{P}^n}(-e)$, $\pi\colon X = \mathbb{P}(\mathcal{E}) \to \mathbb{P}^n$. Denote $E \subseteq X$ by the section with $E|_E \simeq \mathcal{O}_{\mathbb{P}}(-e)$, $E \simeq \mathbb{P}^n$. Let $f\colon X \to Z$ be the contraction of E to a point, which is a contraction of the extremal ray generated by $[C]$, $C \subseteq E$.

$$\begin{tikzcd} X \arrow[r] \arrow[d, "\mathbb{P}^1\text{-bundle}"] & Z = \text{ cone over } \mathbb{P}^n \\ \mathbb{P}^n \end{tikzcd}$$

By adjunction we have $K_E \sim (K_X + E)|_E$. If $L \subseteq E$ is a line, then

$$-n-1 = K_E \cdot L = K_X \cdot L + E \cdot L = K_X \cdot L - e \implies K_X \cdot L = e - n - 1.$$

So,

- $e < n+1 \implies K_X \cdot L < 0$;
- $e = n+1 \implies K_X \cdot L = 0$;
- $e > n+1 \implies K_X \cdot L > 0$.

Write

$$K_X + \alpha E = f^*K_Z.$$

Then

$$\alpha = \frac{e-n-1}{e} = 1 - \frac{n+1}{e}.$$

So if $n+1 \le e$, then Z is not smooth because $\alpha \ge 0$ meaning singularities of Z are worse than terminal singularities (smooth $\implies$ terminal). Note $(K_X + \beta E) \cdot L < 0$ for any $\beta \in (\alpha, 1]$. So $f\colon X \to Z$ is a step of the LMMP for $(X, \beta E)$.

Example 10.16 (Flip)**.** Let $Z \subseteq \mathbb{P}^4$ be defined by $xy - tu = 0$. Note that Z is singular at $p = (0:0:0:0:1)$. Also, Z is the projective cone over $V \simeq \mathbb{P}^1 \times \mathbb{P}^1 \subseteq \mathbb{P}^3$ defined by the same $xy - tu = 0$. Blowing up p gives

$$\begin{tikzcd} & W \supseteq E \simeq \mathbb{P}^1 \times \mathbb{P}^1 \arrow[dl] \arrow[dr] \arrow[dd] & \\ X \arrow[rr, dashed, "\phi"] \arrow[dr, "g"'] & & X^+ \arrow[dl, "g^+"] \\ & Z. & \end{tikzcd}$$

where $W \to X$ and $W \to X^+$ are extremal divisorial contractions induced by the projections

$$\begin{tikzcd} & E \arrow[dl] \arrow[dr] & \\ \mathbb{P}^1 & & \mathbb{P}^1. \end{tikzcd}$$

Note that X, X^+ are smooth, $X \to Z$ and $X^+ \to Z$ are extremal small contractions. Here K_X is Cartier, so we can take g^*K_Z and $(g^+)^*K_Z$ Since g, g^+ are small,

$$g^*K_Z = K_X, \quad (g^+)^*K_Z = K_{X^+} \implies K_X \cdot C = 0 = K_{X^+} \cdot C^+.$$

So $X \to Z \leftarrow X^+$ is a K_X-*flop*. Now take an ample divisor A^+ on X^+, and $A; \phi_*^{-1}A^+$. Then one can show $A \cdot C < 0$. So $X \to Z$ is a $K_X + \epsilon A$-flipping contraction ($\epsilon > 0$ small). And $X \to Z \leftarrow X^+$ is the flip, which is a step of the LMMP on $(X, \epsilon A)$. Now $W \to X$ and $W \to X^+$ are both first steps of LMMP on $(W, 0)$.

10.17. Let (X, B) be a projective log canonical pair, and $f\colon X \to Z$ a flipping contraction (f is an extremal contraction, $(K_X + B) \cdot C < 0$, $\forall C \subseteq X$ contracted by f). How to construct the flip of f?

$$\begin{tikzcd} K_X + B & X \arrow[rr, dashed] \arrow[dr] & & X^+ \arrow[dl] & K_{X^+} + B^+ \\ & & Z & & \end{tikzcd}$$

Consider

$$\mathcal{R} = \oplus_{m \geq 0} f_* \mathcal{O}_X(m(K_X + B)).$$

This is a graded $\mathcal{O}_Z$-algebra. Assume $\mathcal{R}$ is a finitely generated $\mathcal{O}_Z$-algebra: for any open affine $U \subseteq Z$, $\mathcal{R}(U)$ is a finitely generated $\mathcal{O}_Z(U)$-algebra. Then it turns out

$$X^+ = \operatorname{Proj} \mathcal{R}.$$

Conversely, if the flip exists, then one can show $\mathcal{R}$ is a finitely generated $\mathcal{O}_Z$-algebra and $X^+ = \operatorname{Proj} \mathcal{R}$. In particular, the flip is unique if it exists.

10.18 (Relation between outcomes of the LMMP)**.** Let (X, B) be a projective log canonical pair. Assume we can run an LMMP on (X, B). The outcome is not necessarily unique.

Example 10.19. Let $X \to \mathbb{P}^2$ be the blowup of a point, $B = 0$.

$$\begin{tikzcd} & X \arrow[dl] \arrow[dr, "\mathbb{P}^1\text{-bundle}"] & \\ \mathbb{P}^2 & & \mathbb{P}^1 \end{tikzcd}$$

Run LMMP on $(X, 0)$. We have two choices:

- $X \to \mathbb{P}^1$ is a Mori fibre space, $Y = X$ outcome; or
- first do the contraction $X \to \mathbb{P}^2$ and get the Mori fibre space $\mathbb{P}^2 \to \mathrm{pt}$.

Remark 10.20. When X is a smooth surface, $B = 0$, $\kappa(K_X) \geq 0$, it is well-known the outcome is unique.

Now in general assume (X, B) is projective, Kawamata log terminal, and assume (Y_1, B_{Y_1}), (Y_2, B_{Y_2}) are minimal model outcomes of LMMP on (X, B).

Theorem 10.21 (Kawamata)**.** *Then (Y_1, B_{Y_1}) and (Y_2, B_{Y_2}) are connected by flops:*

$$\begin{tikzcd} & (X, B) \arrow[dl, "LMMP"'] \arrow[dr, "LMMP"] & \\ (Y_1, B_{Y_1}) \arrow[rr, dashed, "\text{sequence of flops with respect to } K_{Y_i} + B_{Y_i}"'] & & (Y_2, B_{Y_2}) \end{tikzcd}$$

If we have two Mori fibre spaces as outcomes, then relations are more complicated.

Example 10.22. Consider any birational map $\phi\colon \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$. Pick a smooth projective X with

$$\begin{tikzcd} & X \arrow[dl, "\text{bir}"', "f"] \arrow[dr, "\text{bir}", "g"'] & \\ \mathbb{P}^2 \arrow[rr, dashed, "\phi"'] & & \mathbb{P}^2. \end{tikzcd}$$

Write

$$K_X + f^*K_{\mathbb{P}^2} + \sum e_i E_i, \quad E_i \text{ exceptional curves of } f.$$

Then $\mathbb{P}^2$ is smooth $\implies$ $\mathbb{P}^2$ has terminal singularities $\implies$ $e_i > 0, \forall i$ $\implies$ running LMMP on K_X over $\mathbb{P}^2$, ends with $\mathbb{P}^2$, because $sume_iE_i$ is effective and exceptional. Similarly, running LMMP on K_X over the other $\mathbb{P}^2$ ends with $\mathbb{P}^2$. So both $\mathbb{P}^2$ are outcomes on LMMP on K_X. But $\phi\colon \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ can be complicated.

11 Cone and contraction, vanishing, nonvanishing, and base point freeness

The cone theorem allows us to perform the very first step of the LMMP, that is, to identify a negative extremal ray and to contract it. The formulation of cone theorem was mainly inspired by Mori's work. However, the proof we present came from an entirely different set of ideas conceived and developed by Shokurov and Kawamata (except the existence of rational curves which relies on Mori's original work). These latter ideas proved to be fundamental, far beyond the proof of the cone theorem.

Theorem 11.1 (Cone and contraction)**.** *Let (X, B) be a klt pair of dimension d with B rational. Then, there is a countable set of $(K_X + B)$-negative extremal rays $\{R_i\}$ such that*

- $\overline{\mathrm{NE}}(X) = \overline{\mathrm{NE}}(X)_{K_X+B\geq 0} + \sum_i R_i$.
- *R_i can be contracted.*
- *$\{R_i\}$ is discrete in $\overline{\mathrm{NE}}(X)_{K_X+B<0}$.*
- *Each R_i contains the class of some rational curve C_i satisfying*

$$-2d \leq (K_X + B) \cdot C_i.$$

- *Let $f\colon X \to Y$ be the contraction of a $K_X + B$-negative extremal ray R, and let L be a Cartier divisor on X with $L \cdot R = 0$. Then, there is a Cartier divisor L_Y on Y such that $L \sim f^*L_Y$.*

Let (X, B) be a klt pair. If R is a $K_X + B$-negative extremal ray, then we will see that it is not difficult to find an ample $\mathbb{Q}$-divisor H such that $H + t(K_X + B)$ is nef and that R is the only extremal ray satisfying $(H + t(K_X + B)) \cdot R = 0$. The main idea is to prove that t is a rational number and to prove that $H + t(K_X + B)$ is semi-ample. The latter divisor then defines a contraction of R.

Theorem 11.2 (Base point free)**.** *Let (X, B) be a klt pair with B rational. Suppose that for a Cartier divisor D with is nef, there is some rational number $a > 0$ such that $aD - (K_X + B)$ is nef and big. Then, mD is free for any natural number $m \gg 0$.*

Theorem 11.3 (Rationality)**.** *Let (X, B) be a klt pair with B rational. Let H be an ample Cartier divisor on X. Suppose that $K_X + b$ is not nef. Then,*

$$\lambda = \max\{t > 0 \mid t(K_X + B) + H \text{ is nef}\}$$

is a rational number. Moreover, one can write $\lambda = \frac{a}{b}$ where $a, b \in \mathbb{N}$ and b is bounded depending only on (X, B).

Proof of Theorem 11.1. Step 1. We can assume that $K_X + B$ is not nef. For any nef $\mathbb{Q}$-Cartier divisor D, define

$$F_D := \{c \in \overline{\mathrm{NE}}(X) \mid D \cdot c = 0\}$$

which is an extremal face of $\overline{\mathrm{NE}}(X)$. We will concentrate on those D for which $\dim F_D = 1$. Let $\mathcal{C}$ be the closure of

$$\overline{\mathrm{NE}}(X)_{K_X+B\geq 0} + \sum_D F_D$$

where D run through nef $\mathbb{Q}$-Cartier divisors with $\dim F_D = 1$. We want to prove that $\mathcal{C} = \overline{\mathrm{NE}}(X)$. Suppose that $\mathcal{C} \neq \overline{\mathrm{NE}}(X)$. Then we can find $c \in \overline{\mathrm{NE}}(X) \setminus \mathcal{C}$ and we can find a rational linear function $N_1(X) \xrightarrow{\phi} \mathbb{R}$ which is positive on $\mathcal{C} \setminus \{0\}$ bur negative on c. Since $N_1(X)$ is dual to $N^1(X)$, there is some rational $\mathbb{Q}$-Cartier divisor G which gives ϕ (i.e., ϕ is nothing but intersection with G).

Step 2. Now if $t \gg 0$ then $G - t(K_X + B)$ is positive on $\overline{\mathrm{NE}}(X)_{K_X+B\leq 0} \setminus \{0\}$. Define

$$\gamma := \min\{t > 0 \mid G - t(K_X + B) \text{ is nef on } \overline{\mathrm{NE}}(X)_{K_X+B\leq 0}\}.$$

Note that $G - \gamma(K_X + B)$ is not positive on $\overline{\mathrm{NE}}(X)_{K_X+B\leq 0} \setminus \{0\}$. So there is some $c' \in \overline{\mathrm{NE}}(X)_{K_X+B<0}$ such that $(G - \gamma(K_X + B)) \cdot c' = 0$. It turns out that $G - \gamma(K_X + B)$ is nef, otherwise there is some $c'' \in \overline{\mathrm{NE}}(X)_{K_X+B>0}$ satisfying

$$(G - \gamma(K_X + B)) \cdot c'' \leq 0.$$

This is not possible because $G - \gamma(K_X + B)$ is positive on any point in $[c', c''] \cap \overline{\mathrm{NE}}(X)_{K_X+B=0}$. By construction, $G - \gamma'(K_X + B)$ is ample for some $\gamma' > \gamma$ but very close to γ. Put $H = G - \gamma'(K_X + B)$. We could assume that H is Cartier.

Step 3. Now put

$$\lambda = \max\{t \mid H + t(K_X + B) \text{ is nef}\}$$

which is rational, and put

$$D = H + \lambda(K_X + B)$$

which is nef. From the construction we see that

$$F_D \cap \mathcal{C} = 0.$$

It might happen that $\dim F_D > 1$. We will try to find another D' such that $F_{D'} \subsetneq F_D$ and $\dim F_{D'} = 1$. First take an ample divisor H_1 such that H_1 and $K_X + B$ are linearly independent on F_D. For $s > 0$, define

$$\lambda(s) := \max\{t \mid sD + H_1 + t(K_X + B) \text{ is nef}\}.$$

Since D is numerically trivial on F_D, it follows that $\lambda(s)$ is bounded. Moreover, if $s' > s \gg 0$ then $\lambda(s') \geq \lambda(s)$. By the rationality theorem, $\lambda(s)$ are rational with bounded denominators. Therefore, $\lambda(s)$ is independent of s when $s \gg 0$. In particular, $F_{sD+H_1+\lambda(s)(K_X+B)} \subsetneq F_D$, where the inclusion is strict because H_1 and $K_X + B$ are linearly independent on F_D. Arguing inductively, there is a rational nef divisor D' such that $F_{D'} \subsetneq F_D$ and $\dim F_{D'} = 1$. This contradicts the above assumptions. Therefore, $\mathcal{C} = \overline{\mathrm{NE}}(X)$.

Step 4. Let D be a nef $\mathbb{Q}$-Cartier divisor such that $\dim F_D = 1$ and such that $D = H + t(K_X + B)$ for some ample Cartier divisor H and some rational number $t > 0$. Then by the base point free theorem D is semi-ample. Therefore, the extremal ray F_D can be contracted.

Step 5. We will prove that such F_d cannot accumulate in $\overline{\mathrm{NE}}(X)_{K_X+B<0}$. Suppose that this is not the case, let F_{D_i} has a limit R which is a ray in $\overline{\mathrm{NE}}(X)$ and suppose that $(K_X + B) \cdot R < 0$. We take the D_i to be the form $D_i = H_i + t_i(K_X + B)$ for certain ample Cartier divisors H_i and rational numbers $t_i > 0$. By the argument, for each i, there exists some $s_i \gg 0$ such that if $D'_i = H + s_i S_i + t'_i(K_X + B)$ where H is a fixed ample divisor and t_i is the maximal number making D'_i nef. Then $F_{D'_i} = F_{D_i}$. For each i, let c_i to be an element of $\overline{\mathrm{NE}}(X)$ such that $(K_X + B) \cdot c_i = -1$ and $c_i \in F_{D_i}$. Then $H \cdot c_i = -t'_i(K_X + B) \cdot c_i = t'_i$. On the other hand, t'_i is a rational number with bounded denominator and $H \cdot c_i = t'_i$ is bounded. Thus, there are only finitely possibilities for the $H \cdot c_i$. But if H is general, then this is possible only if there is finitely many c_i. Therefore, $\overline{\mathrm{NE}}(X) = \overline{\mathrm{NE}}(X)_{K_X+B\geq 0} + \sum R_i$ where R_i are the negative extremal rays. We have already proved that each R_i is contractible.

Step 6. The fact that C_i can be chosen to be rational and satisfying $-2d \leq (K_X + B) \cdot C_i$ is proved using very different arguments.

Step 7. Let L, R and $f\colon X \to Y$ be as in the last claim of the theorem. Let D be a nef Cartier divisor such that $F_D = R$ and $D = H + t(K_X + B)$ where H is an ample Cartier divisor. If $a \gg 0$, then $L + aD$ is nef (because D is nef positive on $\overline{\mathrm{NE}}(X)_{K_X+B\geq 0}$ hence $L + aD$ is also positive on it when $a \gg 0$) and $F_{L+aD} = F_D$. By the base point free theorem, for some large m, $m(L + aD)$ and $(m + 1)(L + aD)$ are both base point free and both are pullbacks of Cartier divisor on Y. This implies that $L + aD$ is also the pullback of some Cartier divisor on Y. If we choose a sufficiently divisible, then aD is the pullback of some Cartier divisor on Y. Therefore, L is the pullback of some Cartier divisor on Y. $\square$

Theorem 11.4 (Kawamata-Viehweg vanishing)**.** *Let (X, B) be a klt pair. Let N be an integral $\mathbb{Q}$-Cartier divisor on X such that $N \equiv K_X + B + M$ where M is nef and big. Then,*

$$H^i(X, \mathcal{O}_X(N)) = 0$$

for any $i > 0$.

Theorem 11.5 (Shokurov Nonvanishing)**.** *Let (X, B) be a projective klt pair where B is rational. Let $G \geq 0$ be a Cartier divisor such that $aD + G - (K_X + B)$ is nef and big for some nef Cartier divisor D and rational number $a > 0$. Then,*

$$H^0(X, mD + G) \neq 0$$

for $m \gg 0$.

Remark 11.6. Suppose that S is a smooth prime divisor in a smooth projective variety X. Then we have an exact sequence

$$0 \longrightarrow \mathcal{O}_X(-S) \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{O}_S \longrightarrow 0.$$

Let D be a Cartier divisor on X. Then, we have the exact sequence

$$0 \longrightarrow \mathcal{O}_X(D - S) \longrightarrow \mathcal{O}_X(D) \longrightarrow \mathcal{O}_S(D|_S) \longrightarrow 0,$$

which gives the following exact sequence of cohomologies

$$\begin{aligned} 0 \longrightarrow H^0(X, D - S) \longrightarrow H^0(X, D) \xrightarrow{f} H^0(S, D|_S) \xrightarrow{g} \\ H^1(X, D - S) \xrightarrow{h} H^1(X, D) \longrightarrow H^1(S, D|_S). \end{aligned}$$

It is often very important to prove that $h^0(X, D) \neq 0$. We like to lift a section on S to X. Assume that $h^0(S, D|_S) \neq 0$. In general this does not imply that $h^0(X, D) \neq 0$. One of the best things that could happen is that $h^1(X, D - S) = 0$. If this vanishing holds then f is surjective. In fact, if h is injective, then again f is surjective.

If $h^0(S, D|_S) \neq 0$ and $h^1(X, D - S) = 0$, then this means that S is not a component of $\operatorname{Fix} D$.

Let X be a complex manifold, and T_X its tangent bundle. A Hermitian form h on X is a tensor 2-form, locally can be written as $h = \sum_{jk} h_{jk} dz_j \otimes d\bar{z}_k$, such that

- h is $\mathbb{C}$-linear in the first and $\mathbb{C}$-anti-linear in the second variable;
- $h(u, v) = \overline{h(v, u)}$ for any $u, v \in T_X$.

12 The base point free theorem

Recall the non-vanishing theorem of Shokurov:

Theorem 12.1 (non-vanishing)**.** *Let (X, B) be a projective klt pair, D nef Cartier divisor, $G \geq 0$ Cartier divisor. Assume that $aD + G - (K_X + B)$ is nef and big for some $a \in \mathbb{Q}^{>0}$. Then*

$$H^0(X, mD + G) \neq 0$$

for $m \gg 0$.

Theorem 12.2 (base point free)**.** *Let (X, B) be a projective klt pair., D a nef Cartier divisor. Assume that $aD - (K_X + B)$ is nef and big for some $a \in \mathbb{Q}^{>0}$. Then mD is base point free for $m \gg 0$, $m \in \mathbb{N}$.*

Proof. We apply induction on dimension. The case $\dim X = 1$ is easy (do as exercise). So assume $\dim X \geq 2$. By assumption

$$aD = K_X + B + A,$$

where A is nef and big. By Corollary 9.12, there is a $\mathbb{Q}$-divisor $C \geq 0$ such that for $n \gg 0$ we can write $A = A' + \frac{1}{n}C$, where A' is ample. Since $n \gg 0$, $(X, B + \frac{1}{n}C)$ is klt. Note $aD = K_X + B + \frac{1}{n}C + A'$ is klt plus ample. So replacing (X, B) with $(X, B + \frac{1}{n}C)$, and replacing A with A', can assume A ample. Applying Shokurov non-vanishing with $G = 0$, we get

$$H^0(X, mD) \neq 0, \quad \text{for } m \gg 0.$$

Pick $b \in \mathbb{N}$ such that $H^0(X, bD) \neq 0$.

If $\operatorname{Bs}|bD| = \emptyset$ for each choice of b, then $\operatorname{Bs}|mD| = \emptyset$ for $m \gg 0$, so we are done. Assume $\operatorname{Bs}|bD| \neq \emptyset$ for a choice of b. By the theory of resolution of singularities, there is a log resolution $f\colon Y \to X$ of $(X, B + D)$ such that

$$\begin{cases} f^*bD \sim M + F, \\ \operatorname{Bs}|M| = \emptyset, \\ \operatorname{Fix}|f^*bD| = F, \quad \operatorname{Bs}|f^*bD| = \operatorname{Supp} F. \end{cases}$$

For each $\ell \in \mathbb{N}$, we can write $f^*\ell bD \sim \ell M + \ell F \sim N + \ell F$ where N is reduced with smooth components, by Bertini theorem (N is a general member of $|\ell M|$). We aim to show

$$\operatorname{Supp} F \not\subseteq \operatorname{Bs}|f^*mD|, \quad \forall m \gg 0.$$

Fix $m_0 \gg 0$. We have

$$m_0 D = aD + (m_0 - a)D = K_X + B + (m_0 - a)D + A.$$

By Corollary 9.12, can write

$$f^*((m_0 - a)D + A) = H + E,$$

with H ample, $E \geq 0$ with fixed support, $\operatorname{Supp}(N + \ell F) \subseteq E$ (ℓ as above, fixed). Now

$$\begin{aligned} f^*(m + m_0)D &= f^*(K_X + B + mD + (m_0 - a)D + A) \\ &= f^*(K_X + B + tD) + f^*(m - t)D + f^*((m_0 - a)D + A) \\ &= f^*(K_X + B + tD) + f^*(m - t)D + H + E. \end{aligned}$$

Since (X,B) is klt,

$$f^*(K_X+B)=K_Y+B'_Y-L'$$

such that (Y,B'_Y) is klt, $L'\geq 0$ exceptional over X, B'_Y and L' have no common components. We can choose $t>0$ and modify coefficients of E such that

$$\begin{aligned} f^*(K_X+B+tD)+E &= K_Y+B'_Y-L'+E+\frac{t}{\ell b}N+\frac{t}{b}F \\ &= K_Y+B_Y+S-L \end{aligned}$$

where

$$\begin{cases} (Y,B_Y) \text{ is klt,} \\ S \text{ reduced, irreducible, a component of } F, \\ L\geq 0 \text{ exceptional over } X, \\ B_Y,S,L \text{ have no common components.} \end{cases}$$

Now

$$\begin{aligned} f^*(m+m_0)D+\lceil L\rceil-S &= f^*(K_X+B+tD)+f^*(m-t)D+H+E+\lceil L\rceil-S \\ &= K_Y+B_Y+S-L+f^*(m-t)D+H+\lceil L\rceil-S \\ &= K_Y+B_Y+\lceil L\rceil-L+f^*(m-t)D+H. \end{aligned}$$

By construction,

$$\begin{cases} (Y,B_Y+\lceil L\rceil-L) \text{ klt,} \\ f^*(m-t)+H \text{ ample.} \end{cases}$$

Then by Kawamata-Viehweg vanishing theorem,

$$H^1(Y,f^*(m+m_0)D+\lceil L\rceil-S)=0.$$

Therefore, considering

$$0\longrightarrow \mathcal{O}_Y(-S)\longrightarrow \mathcal{O}_Y\longrightarrow \mathcal{O}_S\longrightarrow 0$$

and tensoring with

$$\mathcal{O}_Y(f^*(m+m_0)D+\lceil L\rceil)$$

we get

$$H^0(Y,f^*(m+m_0)D+\lceil L\rceil)\xrightarrow{\alpha} H^0(S,f^*(m+m_0)D+\lceil L\rceil|_S)\longrightarrow H^1(Y,f^*(m+m_0)D+\lceil L\rceil-S)=0,$$

where α is surjective. On the other hand,

$$f^*(m+m_0)D+\lceil L\rceil|_S=(K_Y+B_Y+S+\lceil L\rceil-L)|_S+(f^*(m-t)D+H)|_S,$$

nef plus effective equals klt plus ample. Therefore, by Shokurov non-vanishing,

$$H^0(S,f^*(m+m_0)D+\lceil L\rceil|_S)\neq 0,\quad m\gg 0.$$

Then α being surjective implies that

$$H^0(Y,f^*(m+m_0)D+\lceil L\rceil)\neq 0 \quad\text{and}\quad S\not\subseteq \mathrm{Bs}\,|f^*(m+m_0)D+\lceil L\rceil|,\ m\gg 0$$

Since $L \geq 0$ is exceptional over X,

$$H^0(Y, f^*(m+m_0)D) \neq 0 \quad \text{and} \quad S \not\subseteq \mathrm{Bs}|f^*(m+m_0)D|,\ m \gg 0$$

and hence

$$H^0(Y, f^*mD) \neq 0 \quad \text{and} \quad S \not\subseteq \mathrm{Bs}|f^*mD|,\ m \gg 0.$$

In particular,

$$\operatorname{Supp} F \not\subseteq \mathrm{Bs}|f^*mD|, \quad \forall m \gg 0.$$

Now pick a large prime number b. Then can assume $H^0(Y, f^*bD) \neq 0$ as above. The above argument shows, there is some n such that

$$\mathrm{Bs}|b^n D| \subsetneq \mathrm{Bs}|bD|,$$

noting that $\mathrm{Bs}|cbD| \subseteq \mathrm{Bs}\, bD$, $\forall c \in N$. Repeating this, can choose n such that

$$\mathrm{Bs}|b^n D| = \emptyset.$$

Choose another large prime number b' and find n' such that

$$\mathrm{Bs}|b'^{n'} D| = \emptyset.$$

Now any $m \gg 0$ can be written as $m = pb^n + qb'^{n'}$ for some $p, q \in \mathbb{Z}^{\geq 0}$. Therefore,

$$\mathrm{Bs}\,|mD| \subseteq \mathrm{Bs}\,|pb^n D| \cup \mathrm{Bs}\,|qb'^{n'} D| \subseteq \mathrm{Bs}\,|b^n D| \cup \mathrm{Bs}\,|qb'^{n'} D| = \emptyset.$$

Hence $\mathrm{Bs}|mD| = \emptyset$, $\forall m \gg 0$. □

Remark 12.3. Let (X, B) be a projective klt pair, $K_X + B$ nef and big. Then $m(K_X + B)$ is base point free for some $m > 0$:

- choose $\ell > 1$ such that $\ell(K_X + B)$ is Cartier.
- $\ell(K_X + B) = (K_X + B) + (\ell - 1)(K_X + B)$ is klt plus nef and big.

Then $n\ell(K_X + B)$ is free for $n \gg 0$. Now pet $m = n\ell$, $n \gg 0$.

This shows that the abundance conjecture holds when (X, B) is klt and $K_X + B$ is big.

Exercise 12.4. Assume (X, B) is projective klt, $K_X + B$ nef, B big. Then $m(K_X + B)$ is free for some $m > 0$.

Example 12.5. The base point free theorem does not hold for log canonical pairs. We construct an example. Let E be an elliptic curve, V projective cone over E. Recall we have

$$\begin{tikzcd} C \arrow[r, hook] \arrow[dr, equal, "\text{isom}"'] & X \arrow[r, "\text{bir}", "f"'] \arrow[d, "\pi"', "\mathbb{P}^1\text{-bundle}"] & V \\ & E & \end{tikzcd}$$

Pick a divisor N on E such that $\deg N = 0$ but $mN \not\sim 0$, $\forall m > 0$. Pick a very ample divisor H on V. Let $A \in |f^*\ell H|$ be general and let $B = C + A$.

Claim: $K_X + B$ is nef and big.

It is enough to show $K_X + B$ is nef because increasing ℓ, can make it big. If $K_X + B$ is not nef, then $(K_X + B) \cdot R < 0$ for some extremal ray R. Then $(K_X + C) \cdot R < 0$ and hence $(K_X + \alpha C) \cdot R < 0$ for some $\alpha \in (0, 1)$. Since $(X, \alpha C)$ is klt, by the cone theorem, R is generated by some curve L with

$$-4 \leq (K_X + \alpha C) \cdot L < 0.$$

Note that $L \neq C$ since $(K_X + C) \cdot C = 0$. So $f^*H \cdot L > 0$ and $A \cdot L = \ell f^*H \cdot L \geq \ell$. But since $\ell \gg 0$,

$$(K_X + B) \cdot L = (K_X + C + A) \cdot L = (K_X + \alpha C) \cdot L + (1 - \alpha)C \cdot L + A \cdot L \geq -4 + \ell > 0,$$

a contradiction.

Now let $D = 2(K_X + B + \pi^*N)$. Then D is nef and big, $D - (K_X + B)$ is nef and big. But mD is not free for any $m > 0$ because

$$mD|_C = 2m(K_X + C + \pi^*N)|_C \sim 2m\pi^*N|_C \not\sim 0,$$

for any $m > 0$.

Remark 12.6 (Effective base point free)**.** Kollár proved the following effective version of Kawamata-Shokurov base point free theorem.

Assume (X, B) is projective klt of dimension d, D nef Cartier, $aD - (K_X + B)$ is nef and big for some $a \in \mathbb{N}$. Then

$$2(d + 2)!(a + d)D$$

is free.

13 The non-vanishing theorem

In this lecture, we prove Shokurov's non-vanishing theorem. First, there are some preparations.

13.1 Riemann-Roch

Let X be a projective variety of dimension d. Let $\mathcal{F}$ be a coherent sheaf. The Euler characteristic of $\mathcal{F}$ is

$$\chi(\mathcal{F}) = h^0(X, \mathcal{F}) - h^1(X, \mathcal{F}) + h^2(X, \mathcal{F}) - \dots$$

For a divisor L, let $\chi(L) = \chi(\mathcal{O}_X(L))$. If L, L' are Cartier divisors and $L \equiv L'$, then $\chi(L) = \chi(L')$.

Theorem 13.1. *Let D be a Cartier divisor, then*

$$\chi(mD) = \frac{(mD)^d}{d!} + \textit{lower degree terms}$$

is a polynomial in m of degree $\leq d$.

Reference: [Kollár-Mori book, 1.36, 2.57].

13.2 Multiplicity of linear systems

Let X be a projective variety, $x \in X$ a smooth closed point, and D a Cartier divisor with $h^0(X, D) > 0$. Changing D linearly, can assume $x \notin \operatorname{Supp} D$. Pick a basis $h_1, \dots, h_n$ of $H^0(X, D)$ ($h_i \in k(X)$, the function field). Each $h \in H^0(X, D)$ is uniquely written as $h = \sum a_i h_i$, $a_i \in k$, the ground field. Note that h_i is regular at x because $\operatorname{Div}(h_i) + D \geq 0$ and $x \notin \operatorname{Supp} D$. Pick local parameters $t_1, \dots, t_d$ at x. Each $h \in H^0(X, D)$ can be described as a power series ϕ_h in terms of $t_1, \dots, t_d$ near x. This follows from the fact: $\mathcal{O}_x$ is the local ring at x, $\mathfrak{m}_x$ the maximal ideal. Since $\mathfrak{m}_x = (t_1, \dots, t_d)$, $\mathfrak{m}_x^l / \mathfrak{m}_x^{l+1}$ generated by monomials of degree l in $t_1, \dots, t_d$.

Now, multiplicity of $\operatorname{Div}(h) + D$ at $x > l \iff \phi_h \in \mathfrak{m}_x^{l+1} \iff \phi_h$ has no term of degree $\leq l$. So, since $\phi_h = \sum a_i \phi_{h_i}$, if $h^0(X, D) > \#\{\text{monomials in } t_1, \dots, t_d \text{ of degree } \leq l\}$, then we can find $h \in H^0(X, D)$ with multiplicity $> l$ at x. There exists $0 \leq D' \sim D$ with multiplicity $> l$ at x, because vanishing of coefficients of each monomial is a linear condition on $H^0(X, D)$.

Note

$$\#\{\text{monomials of degree } \leq l\} = \frac{(l+1)^d}{d!} + \text{lower degree terms}.$$

13.3 The non-vanishing theorem

Theorem 13.2. *Let (X, B) be a projective klt pair, D nef Cartier divisor, $G \geq 0$ Cartier divisor, $aD + G - (K_X + B)$ nef and big for some $a \in \mathbb{Q}^{>0}$. Then $h^0(X, mD + G) > 0$, $\forall m \gg 0$.*

Proof. We apply induction on $d = \dim X$. Case $d = 1$: do an exercise. Assume $d \geq 2$.

Step 1: Reduce to smooth case. By Corollary 9.12, can write

$$f^*(aD + G - (K_X + B)) = H + E'$$

where H is ample, $E' \geq 0$ with small coefficients.

Moreover, (X,B) klt $\implies$ can write $f^*(K_X+B) = K_X + B_Y - G'$, such that (Y, B_Y) klt, $G' \geq 0$ Cartier, exceptional over X. Then

$$\begin{aligned} f^*(aD + G - (K_X+B)) &= f^*aD + f^*G - (K_Y + B_Y - G') \\ &= f^*aD + f^*G + G' - (K_Y + B_Y). \end{aligned}$$

Thus

$$f^*aD + f^*G + G' - (K_Y + B_Y + E') = H$$

where E' can be chosen such that $(Y, B_Y + E')$ is klt. Thus replacing (X,B), D, G with (Y, B_Y+E'), f^*D, $f^*G + G'$, can assume X smooth, D, G with simple normal crossing singularities.

Step 2. Assume $D \equiv 0$. By assumption, $aD + G = K_X + B+$ nef and big. Then $D \equiv 0 \implies$

$$\begin{cases} mD + G = K_X + B + \text{nef and big}, \forall m \in N, \\ G = K_X + B + \text{nef and big}. \end{cases}$$

Then by Kawamata-Viehweg vanishing,

$$\begin{aligned} h^i(X, mD+G) &= 0, \quad \forall i > 0, m \in \mathbb{N} \\ h^i(X, G) &= 0, \quad \forall i > 0. \end{aligned}$$

Hence

$$h^0(X, mD+G) = \chi(mD+G) = \chi(G) = h^0(X,G) > 0.$$

So we are done in the case $D \equiv 0$. From now on we assume $D \not\equiv 0$.

Step 3. Put $A \coloneqq aD + G - (K_X+B)$. We can assume A ample by Step 1. For $a \leq n \in \mathbb{N}$,

$$\begin{aligned} (nD + G - (K_X+B))^d &= ((n-a)D + aD + G - (K_X+B))^d \\ &= ((n-a)D + A)^d \\ &= ((n-a)D)^d + d((n-a)D)^{d-1} \cdot A + \cdots + d(n-a)D \cdot A^{d-1} + A^d \\ &\geq d(n-a)D \cdot A^{d-1}. \end{aligned}$$

Since A ample, $D \not\equiv 0$ nef, $D \cdot A^{d-1} > 0$. Pick $k \in \mathbb{N}$ such that kA and $k(K)X_B$ are Cartier. By Serre vanishing and by Riemann-Roch, for fixed $n \geq a$, $l \gg 0$,

$$\begin{aligned} h^0(X, kl(nD + G - (K_X+B))) &= \chi(kl(nD + G_(K_X+B))) \\ &= \frac{(kl(nD+G-(K_X+B)))^d}{d!} + \text{lower degree terms} \end{aligned}$$

is a polynomial in l. So taking n large enough, can ensure

$$h^0(X, kl(nD+G-(K_X+B))) > \#\{\text{monomials of degree } \leq 2kld \text{ in } d \text{ varieties}\}.$$

By Section 13.2, for a closed $x \in X \setminus \operatorname{Supp} G$, there exists $0 \leq N \sim kl(nD + G - (K_X+B))$ with multiplicity $> 2kld$ at x. Put $L_n = \frac{1}{kl}N$. Then

$$0 \leq L_n \sim_{\mathbb{Q}} nD + G - (K_X+B)$$

and multiplicity of L_n at x is $> 2d$. In particular, $(X, B + L_n)$ is not log canonical at x: just consider the blowup of X at x. Now take a log resolution $g\colon W \to X$. For $m \gg 0$ and $t \in \mathbb{Q}^{>0}$, can write

$$\begin{aligned} g^*(mD+G) &= g^*(K_X + B + tL_n + mD + G - (K_X+B) - tL_n) \\ &\sim_{\mathbb{Q}} g^*(K_X + B + tL_n + mD + G - (K_X+B) - tnD - tG + t(K_X+B)) \\ &= g^*(K_X + B + tL_n + (m-tn)D + (1-t)G - (1-t)(K_X+B)) \\ &= g^*(K_X + B + tL_n + (m - tn - (1-t)a)D + (1-t)aD + (1-t)G - (1-t)(K_X+B)) \\ &= g^*(K_X + B + tL_n) + g^*(m - tn - (1-t)a)D + g^*(1-t)A \\ &\sim_{\mathbb{Q}} g^*(K_X + B + tL_n) + H + E', \end{aligned}$$

where H is ample, $E' \geq 0$. We can choose t, H, E' such that

$$g^*(K_X + B + tL_n) + H + E' = K_Y + B_Y + S - G' + H$$

and

$$\begin{cases} (Y, B_Y) \text{ klt}, \\ S \text{ reduced, irreducible, not a component of } B_Y, \\ G' \geq 0 \text{ Cartier, exceptional over } X, \\ H \text{ ample.} \end{cases}$$

We then get

$$g^*mD + g^*D + G' \sim_{\mathbb{Q}} K_W + B_W + S + H.$$

Then

$$(g^*mD + g^*G + G')|_S \sim_{\mathbb{Q}} (K_W + B_W + S)|_S + H|_S,$$

so by induction,

$$h^0(S, (g^*mD + g^*G + G')|_S) > 0, \quad \forall m \gg 0.$$

On the other hand,

$$g^*mD + g * G + G' - S \sim_{\mathbb{Q}} K_W + B_W + H$$

so by Kawamata-Viehweg vanishing,

$$h^1(W, g^*mD + g^*D + G' - S) = 0.$$

Consider the short exact sequence

$$0 \longrightarrow \mathcal{O}_W(g^*mD + g^*G + G' - S) \longrightarrow \mathcal{O}_W(g^*mD + g^*G + G') \longrightarrow \mathcal{O}_S((g^*mD + g^*G + G')|_S) \longrightarrow 0$$

which induces

$$H^0(W, g^*mD + g^*G + G') \to H^0((S, g^*mD + g^*G + G')|_S) \to H^1(W, g^*mD + g^*G + G' - S) = 0.$$

We deduce that

$$h^0(W, g^*mD + g^*G + G') > 0, \quad \forall m \gg 0$$

and hence

$$h^0(X, mD + G) > 0, \quad \forall m \gg 0.$$

□

Remark 13.3. We have given an almost complete proof of

- the cone and contraction theorem,
- the base Point free theorem,
- the non-vanishing theorem,

except that we didn't prove the rationality theorem. To run the MMP we need existence of flips. In dimension 2, flips does not appear, so we can run MMP on any projective klt pair.

Remark 13.4. We discussed the above theorems only for projective klt pairs. Assume (X, B) is klt and $X \to Z$ is a projective morphism, but X may not be projective. Then above theorems also hold for (X, B) over Z. We only need some small changes in the proofs. For example, if D is Cartier and nef over Z, and if $aD - (K_X + B)$ is nef and big over Z, then mD is base point free over Z, $\forall m \gg 0$. For more details, see [Kollár-Mori, section 3.6].

14 D-flips and finite generation

14.1 Finite generation problem

Let $f\colon X \to Z$ be a contraction of normal varieties, D a $\mathbb{Q}$-divisor on X. Define

$$\mathcal{R}(D/Z) = \oplus_{m\geq 0} f_*\mathcal{O}_X(\lfloor mD\rfloor)$$

which is a graded $\mathcal{O}_Z$-algebra: for any open $U \subseteq Z$, $\mathcal{R}(D/Z)(U)$ is a graded $\mathcal{O}_Z(U)$-algebra.

Question 14.1. *When is $\mathcal{R}(D/Z)$ a finitely generated $\mathcal{O}_Z$-algebra? That is, Z can be covered by open affine U such that $\mathcal{R}(D/Z)(U)$ is a finitely generated $\mathcal{O}_Z$-algebra.*

When $Z = \mathrm{pt}$, we write $\mathcal{R}(D)$ instead of $\mathcal{R}(D/Z)$. In this case, $\mathcal{R}(D)$ is a k-algebra (k = ground field).

Exercise 14.2 (Truncation principle)**.** Let $I \in \mathbb{N}$. Show that $\mathcal{R}(D/Z)$ is a finitely generated $\mathcal{O}_Z$-algebra if and only if $\mathcal{R}(ID/Z)$ is a finitely generated $\mathcal{O}_Z$-algebra.

More generally, given a Noetherian graded integral domain $R = \oplus_{m\geq 0} R_m$, then R is a finitely generated R_0-algebra if and only if $\oplus_{m\geq 0} R_{mI}$ is a finitely generated R_0-algebra.

Lemma 14.3. *Let $X \xrightarrow{f} Y \xrightarrow{g} Z$ be contractions of normal varieties. Then $\mathcal{R}(D/Z)$ is a finitely generated $\mathcal{O}_Z$-algebra if and only if $\mathcal{R}(f^*D/Z)$ is a finitely generated $\mathcal{O}_Z$-algebra.*

Proof. By the truncation principle, we can assume D is Cartier. For each open $U \subseteq Z$,

$$\mathcal{R}(D/Z)(U) \simeq \oplus_{m\geq 0} H^0(g^{-1}U, mD) \simeq \oplus_{m\geq 0} H^0(f^{-1}g^{-1}U, mf^*D) \simeq \mathcal{R}(f^*D/Z)(U).$$

So the claim follows. ☐

Lemma 14.4 (Zariski)**.** *Let $h\colon X \to Z$ be a contraction of normal varieties, D a divisor on X. If D is semi-ample over Z, then $\mathcal{R}(D/Z)$ is a finitely generated $\mathcal{O}_Z$-algebra.*

Proof. Here D is semi-ample over Z means, ID is free over Z for some $I \in \mathbb{N}$. And D free over Z means: Z covered by open affine $U \subseteq Z$ such that $D|_{f^{-1}U}$ free.

Can replace Z with such U, so assume Z affine and D free. Now D defines a contraction $f\colon X \to Y$ such that h is factored as $X \xrightarrow{f} Y \to Z$ and a divisor H on Y such that $D \sim f^*H$ for some Cartier divisor H on Y ample over Z.

By the truncation principle can assume H very ample over Z. Also can assume $H^i(Y, mH) = 0$, $\forall i > 0, m > 0$. By Lemma 14.4, enough to show $\mathcal{R}(H/Z)$ is a finitely generated $\mathcal{O}_Z$-algebra. Enough to show: $\mathcal{R}(H/Z) := \mathcal{R}(H/Z)(Y)$ is a finitely generated $\mathcal{A} := \mathcal{O}_Z(Z)$-algebra. Note that there exists a closed embedding $i\colon Y \hookrightarrow \mathbb{P}^n_Z/Z$ for some n, such that $H \sim i^*L$, where L is the pullback of some hyperplane under $\mathbb{P}^n_Z = \mathbb{P}^n \times Z \to \mathbb{P}^n$.

Consider

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^n_Z}(mL) \otimes \mathcal{I}_Y \longrightarrow \mathcal{O}_{\mathbb{P}^n_Z}(mL) \longrightarrow \mathcal{O}_Y(mL|_Y) \simeq \mathcal{O}_Y(mH) \longrightarrow 0$$

where $\mathcal{I}_Y$ is the ideal sheaf of Y, which gives

$$H^0(\mathbb{P}^n_Z, mL) \longrightarrow H^0(Y, mH) \longrightarrow H^1(\mathbb{P}^n_Z, \mathcal{O}_{\mathbb{P}^n_Z}(mL) \otimes \mathcal{I}_Y) = 0$$

for $m \gg 0$, by Serre vanishing. So enough to show $\mathcal{R}(L/Z)$ is a finitely generated $\mathcal{O}_Z$-algebra. This follows from the fact:

$$\mathcal{R}(L/Z) \coloneqq \mathcal{R}(L/Z)(\mathbb{P}^n_Z) = \oplus_{m \geq 0} H^0(\mathbb{P}^n_Z, mL) \simeq A[t_0, \ldots, t_n]$$

where t_i are variables and $A = \mathcal{O}_Z(Z)$ (See Hartshorne, Chapter II, Proposition 5.13). □

Definition 14.5 (D-flips)**.** Let $f \colon X \to Z$ be a contraction of normal varieties, D a $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor on X. Then f is a *D-flipping contraction* if

- f is small, birational, extremal;
- $-D$ ample over Z.

A D-flip is a diagram

$$\begin{tikzcd} X \arrow[rr, dashed, "\varphi"] \arrow[dr, "f"'] & & X^+ \arrow[dl, "f^+"] \\ & Z & \end{tikzcd}$$

such that

- f^+ small birational contraction,
- X^+ normal,
- $D^+ \coloneqq \varphi_* D$ ample over Z.

Theorem 14.6. *Under the notation of Definition 14.5 we have: The D-flip exists if and only if $\mathcal{R}(D/Z)$ is a finitely generated $\mathcal{O}_Z$-algebra.*

Proof. ($\Longrightarrow$) $\mathcal{R}(D/Z) \simeq \mathcal{R}(D^+/Z)$ because f, f^+ are small. Note that $\mathcal{R}(D^+/Z)$ is a finitely generated $\mathcal{O}_Z$-algebra by Lemma 14.4.

($\Longleftarrow$) Can assume Z affine, say $Z = \operatorname{Spec} A$, and that $\mathcal{R}(D/Z) \coloneqq \mathcal{R}(D/Z)(Z)$ is a finitely generated A-algebra. Can find $I \in \mathbb{N}$ such that $\mathcal{R}(ID/Z)$ is generated by degree 1 elements as an A-algebra. Replacing D with ID, can assume $\mathcal{R}(D/Z)$ is generated by degree 1 elements.

Put $X^+ = \operatorname{Proj} \mathcal{R}(D/Z)$ and $f^+ \colon X^+ \to Z$ the associated morphism, and $\mathcal{O}_{X^+}(1)$ the associated invertible sheaf, which is ample over Z (See Hartshorne, pp. 160–161). By Remark 14.7, X^+ is a normal variety.

Note that f^+ is birational: $f|_{f^{-1}V}$ is an isomorphism where V is the smooth locus of Z. So $f^+|_{f^{-1}V}$ is also an isomorphism.

We show f^+ is small. Assume not, say f^+ contracts a prime divisor $E \subseteq X^+$. Let L be the divisor corresponding to $\mathcal{O}_{X^+}(1)$. Then $\mathcal{O}_{X^+}(mL) \subsetneq \mathcal{O}_{X^+}(mL + E)$. Since L is ample over Z,

$$f^+_* \mathcal{O}_{X^+}(mL) \subsetneq f^+_* \mathcal{O}_{X^+}(mL + E)$$

for some $m \gg 0$, by [Hartshorne, Chapter II, 5.15]. Now E is exceptional over Z, so

$$f^+_* \mathcal{O}_{X^+}(mL + E) \subseteq \mathcal{O}_Z(mL_Z), \quad L_Z = f^+_* L.$$

But f^+ is an isomorphism over V, the smooth locus of Z, so can choose L such that $L_Z + D$, so $\mathcal{O}_Z(mL_Z) = \mathcal{O}_Z(mf_* D)$. This contradicts

$$f^+_* \mathcal{O}_{X^+}(m) = f_* \mathcal{O}_X)(mD) = \mathcal{O}_Z(mf_* D) = \mathcal{O}_Z(mL_Z), \quad m \gg 0.$$

Then

$$\begin{tikzcd} X \arrow[rr, dashed, "\varphi"] \arrow[dr, "f"'] & & X^+ \arrow[dl, "f^+"] \\ & Z & \end{tikzcd}$$

is the required D-flip. $\square$

Remark 14.7. Let Z be a normal variety and $\mathcal{R}$ a finitely generated graded $\mathcal{O}_Z$-algebra with the degree 0 piece $\mathcal{R}_0 = \mathcal{O}_Z$. Let $I \in \mathbb{N}$, and let $\mathcal{R}^{[I]}$ be the subalgebra of $\mathcal{R}$ consisting summands of degree divisible by I, i.e., the piece of degree $0, I, 2I, \ldots$. The injection $\mathcal{R}^{[I]} \to \mathcal{R}$ induces a rational map

$$\phi \colon \operatorname{Proj} \mathcal{R} \dashrightarrow \operatorname{Proj} \mathcal{R}^{[I]}$$

over Z. By replacing Z with an open affine subset, from now on we assume that Z is affine, and instead of the above sheaves we consider the corresponding algebras $R = \mathcal{R}(Z)$ and $R^{[I]} = \mathcal{R}^{[I]}(Z)$. Now, ϕ is actually a morphism. Indeed, if $P \in \operatorname{Proj} R$, then $P \cap R^{[I]} \in \operatorname{Proj} R^{[I]}$ because $P \cap \mathcal{R}^{[I]}$ does not contain all elements of $P^{[I]}$ of positive degree since $\alpha \in R$ implies that $\alpha^I \in R^{[I]}$.

Moreover, we show that ϕ is locally an isomorphism. If $\alpha \in R^{[I]}$ has positive degree nI, then the induced localised map $R^{[I]}_{(\alpha)} \to R_{(\alpha)}$ is again injective. In is actually, also surjective. Indeed, let $\frac{\beta}{\alpha^r}$ be an element of $R_{(\alpha)}$. Then, by definition, $\deg \beta = \deg \alpha^r = rnI$. So, $\beta \in R^{[I]}_{(\alpha)}$ hence $\frac{\beta}{}a^r$ is inside $R^{[I]}_{(\alpha)}$. Now let $\alpha_1, \ldots, \alpha_l$ be elements of $R^{[I]}_{(\alpha)}$ which generate $R^{[I]}_{(\alpha)}$ as an R_0-algebra where $R_0 = \mathcal{O}_Z(Z)$.

Let $\mathcal{I}$ be the ideal of R consists of all elements of positive degree, $\mathcal{I} \coloneqq \mathcal{I} \cap R^{[I]}$, and $\mathcal{J}$ the ideal of R generated by the $\alpha_1, \ldots, \alpha_l$. Then, $\mathcal{I} = \sqrt{\mathcal{I}^{[I]} R} \subseteq \sqrt{\mathcal{J}} \subseteq \mathcal{I}$. So, the principal open sets $D_+(\alpha_i) \subseteq \operatorname{Proj} R$ and $D^{[I]}_+(\alpha_i) \subseteq \operatorname{Proj} R^{[I]}$ defined by the α_i cover $\operatorname{Proj} R$ and $\operatorname{Proj} R^{[I]}$. Now the isomorphisms $R^{[I]}_{(\alpha_i)} \to R_{(\alpha_i)}$ imply that

$$\phi_i \colon D_+(\alpha_i) \to D^{[I]}_+(\alpha_i)$$

are isomorphisms hence ϕ itself is an isomorphism.

Let $\mathcal{S}$ be the graded $\mathcal{O}_Z$-algebra whose degree n summand is the degree nI summand of $\mathcal{R}^{[I]}$, and multiplication in $\mathcal{S}$ is the one induced by $\mathcal{R}^{[I]}$. Then, one can see that $\operatorname{Proj} \mathcal{S}$ is isomorphic to $\operatorname{Proj} cR^{[I]}$ as schemes over Z.

If $\mathcal{R} = \mathcal{R}(X/Z, D)$ where D is a $\mathbb{Q}$-divisor on some normal variety X projective over Z, then $\mathcal{S} = \mathcal{R}(X/Z, ID)$. Moreover, in this case, $\operatorname{Proj} \mathcal{R}(X/Z, D)$ is a normal variety: we assume Z is affine and that $D \geq 0$; note that since $R(X/Z, D)$ is an integral domain, $\operatorname{Proj} R(X/Z, D)$ is an integral scheme with function field $R(X/Z, D)_{(0)}$. If α is any homogeneous element of degree l, then we show that $R(X/Z, D)_{(\alpha)}$ is integrally closed in $R(X/Z, D)_{(0)}$. Assume that $\frac{\beta}{\gamma} \in R(X/Z, D)_{(0)}$ satisfying an equation

$$\left(\frac{\beta}{\gamma}\right)^n + \frac{\lambda_1}{\alpha^{r_1}} \left(\frac{\beta}{\gamma}\right)^{n-1} + \cdots + \frac{\lambda^n}{\alpha^{r_n}} = 0$$

where $\frac{\lambda_i}{\alpha^{r_i}} \in R(X/Z, D)_{(\alpha)}$. Let $r = \max r_i$. Multiplying the equation by α^{nr}, and replacing $\frac{\beta\alpha^r}{\gamma}$ by θ and replacing $\lambda_i \alpha^{ir - r_i}$ by ρ_i we get an equation

$$\theta^n + \rho_1 \theta^{n-1} + \cdots + \rho_n = 0$$

and it is enough to prove that $\theta \in R(X/Z, D)$ and that it has degree rl, i.e., $\theta \in H^0(X, rlD)$. Let P be a prime divisor on X. Then, from the equation we can estimate

$$n\mu_P(\theta) = \mu_P(\theta^n) \geq \min\{\mu_P(\rho_i + (n-i)\mu_P(\theta)\}$$

where μ_P stand for the multiplicity. If the minimum is attained at $\mu_P(\rho_j) + (n-j)\mu_P(\theta)$, then since $\rho_i \in H^0(X, rljD)$, $\mu_P(\rho_j) \geq rlj\mu_P(D)$. Thus,

$$j\mu_P \geq lrj\mu_P(D).$$

Therefore, $\theta \in H^0(X, rlD)$ and we are done.

Remark 14.8. If a D-flip exists, then it is unique. By the theorem the flip is determined by $\mathcal{R}(D/Z)$.

Definition 14.9. Let (X, B) be a log canonical pair. Let $f\colon X \to Z$ be a contraction of normal varieties, $K_X + B$ big over Z. The log canonical model of (X, B) over Z exists if there is a diagram

$$\begin{tikzcd} X \arrow[rr, dashed, "\varphi"] \arrow[dr, "f"'] & & Y \arrow[dl, "g"] \\ & Z & \end{tikzcd}$$

where

- φ^{-1} does not contract divisors;
- $K_Y + B_Y \coloneqq \varphi_*(K_X + B)$ is ample over Z;
- $\alpha^*(K_X + B) \geq \beta^*(K_Y + B_Y)$ for any common resolution

$$\begin{tikzcd} & W \arrow[dl, "\alpha"'] \arrow[dr, "\beta"] & \\ X \arrow[dr, "f"'] & & Y \arrow[dl, "g"] \\ & Z & \end{tikzcd}$$

Theorem 14.10. *Let (X, B) be a lc pair, $X \to Z$ a contraction of normal varieties, $K_X + B$ big over Z. Then the log canonical model of (X, B) over Z exists if and only if $\mathcal{R}(K_X + B/Z)$ is a finitely generated $\mathcal{O}_Z$-algebra.*

Proof. ($\implies$) Take a common resolution as in Definition 14.9:

$$\begin{tikzcd} & W \arrow[dl, "\alpha"'] \arrow[dr, "\beta"] & \\ X \arrow[dr, "f"'] & & Y \arrow[dl, "g"] \\ & Z & \end{tikzcd}$$

Then $\alpha^*(K_X+B)-\beta(K_Y+B_Y)$ is effective and exceptional over Y. So,

$$\mathcal{R}(I(K_X+B)/Z)\simeq\mathcal{R}(I\alpha^*(K_X+B)/Z)\simeq\mathcal{R}(I\beta^*(K_Y+B_Y)/Z)\simeq\mathcal{R}(I(K_Y+B_Y)/Z).$$

Now $\mathcal{R}(K_Y+B_Y/Z)$ is a finitely generated $\mathcal{O}_Z$-algebra because K_Y+B_Y is ample over Z, hence $I(K_Y+B_Y)$ is very ample over Z, so we can apply Lemma 14.4. Thus $\mathcal{R}(K_X+B/Z)$ is a finitely generated $\mathcal{O}_Z$-algebra.

($\Longleftarrow$) Can assume Z is affine, because the log canonical model is unique if it exists. There exists some $I\in\mathbb{N}$, such that $\mathcal{R}(I(K_X+B))$ is generated by degree one elements. Let $Y=\operatorname{Proj}\mathcal{R}(I(K_X+B))$, and $Y\to Z$ the associated morphism, $\mathcal{O}_Y(1)$ the associated invertible sheaf which is ample over Z. Take a resolution $\alpha\colon W\to X$ such that $\alpha^*I(K_X+B)=M+F$, where M is free and $F=\operatorname{Fix}|\alpha^*I(K_X+B)|$. Since $\mathcal{R}(I(K_X+B)/Z)$ is generated by degree one elements,

$$mF=\operatorname{Fix}|\alpha^*mI(K_X+B)|,\quad\forall m.$$

Thus $\mathcal{R}(I(K_X+B)/Z)\simeq\mathcal{R}(M/Z)$. Then M defines a contraction $\beta\colon W\to Y\simeq\operatorname{Proj}\mathcal{R}(M/Z)$. Then we get an induced birational map

$$\begin{tikzcd} X \arrow[rr, dashed, "\varphi"] \arrow[dr] & & Y \arrow[dl] \\ & Z & \end{tikzcd}$$

such that φ does not contract divisors (do as an exercise). So $K_Y+B_Y=\varphi(K_X+B)=\beta_*M$ is ample over Z, and $\mathcal{O}_Y(1)$ is the sheaf associated to the divisor $I(K_Y+B_Y)$. Therefore, Y is the log canonical model of (X,B) over Z. □

Remark 14.11. (1) In general it is expected that log canonical models of lc pairs always exist, but this is known only when $\dim X\le 4$.

(2) We can replace K_X+B big over Z with K_X+B pseudo-effective over Z and still expect that $\mathcal{R}(K_X+B/Z)$ finitely generated. In this case, we can also defined the log canonical model as $Y:=\operatorname{Proj}\mathcal{R}(K_X+B/Z)$. We get a diagram

$$\begin{tikzcd} X \arrow[rr, dashed, "\varphi"] \arrow[dr] & & Y \arrow[dl] \\ & Z & \end{tikzcd}$$

But φ is birational only when K_X+B is big over Z.

(3) When (X,B) is klt, $\mathcal{R}(K_X+B/Z)$ is always finitely generated and so a log canonical model exists (BCHM). In particular, flips exists for K_X+B-negative flipping contraction for klt pairs (X,B).

Remark 14.12. Let $f\colon X\to Z$ be a contraction of normal varieties, D a $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor on X. In general $\mathcal{R}(D/Z)$ is not always finitely generated. In fact when D is nef and big over Z, it is well-known that $\mathcal{R}(D/Z)$ is finitely generated if and only if D is semi-ample over Z. We saw in Example 12.5 an example of a nef and big divisor D which is not semi-ample.

15 Existence of flips and extension theorems

Definition 15.1 (dlt pairs)**.** Let (X,B) be a pair, $\varphi\colon W\to X$ a log resolution. Write $K_W+B_W=\varphi^*(K_X+B)$. If a component D of B_W has coefficient ≥ 1, we call $\varphi(D)$ a *non-klt centre* of (X,B). Union of all non-klt centres is denoted $\mathrm{Nklt}(X,B)$.

We say that (X,B) is *divisorial log terminal (dlt)* if (X,B) is lc, and it is log smooth near generic point of each non-klt centre.

Fact: (X,B) is dlt if and only if there is a log resolution $\varphi\colon W\to X$ such that $\operatorname{coeff}_D B_W<1$ for any prime divisor $D\subseteq W$ which is exceptional over X.

Example 15.2. (1) (X,B) log smooth, then dlt.

(2) (X,B) klt, then dlt.

(3) $(\mathbb{P}^2, B=\text{nodal curve})$ lc, not dlt.

(4) $(\mathbb{P}^2, B=B_1+\frac{1}{2}B_2)$ is dlt, where B_1 is a line, B_2 is a nodal curve and B_1 intersects B_2 transversally (not passing through singularities of B_2).

Fact: Let (X,B) be dlt, S component of B with coefficient 1. Then S is normal. ([Kollár-Mori, Corollary 5.52]).

Definition 15.3 (plt pairs)**.** A dlt pair (X,B) is *purely log terminal (plt)* if

- no two components of $\lfloor B\rfloor$ intersects; or equivalently
- its only non-klt centres are the components of $\lfloor B\rfloor$.

Example 15.4. (1) $(\mathbb{A}^2, \text{intersecting lines})$ dlt, not plt.

(2) $(\mathbb{A}^2, \text{parallel lines})$ plt.

Remark 15.5. Let (X,B) be a projective dlt pair. It is not hard to show: running MMP on (X,B) preserves the dlt condition. One of the most convenient settings for running MMP is when (X,B) is $\mathbb{Q}$-factorial dlt.

15.1 Adjunction for dlt pairs

Let (X,B) be a dlt pair, S a component of $\lfloor B\rfloor$. We know that S is normal. There is an adjunction formula:

$$(K_X+B)|_S=K_S+B_S$$

for some uniquely determined boundary divisor B_S.

Explanation: take a log resolution $\varphi\colon W\to X$, and write $K_W+B_W=\varphi(K_X+B)$. Assume $T\subseteq W$ is the birational transform of S. We have an adjunction formula

$$(K_W+B_W)|_T=(K_W+T+B_W-T)|_T=K_T+(B_W-T)|_S=K_T+B_T.$$

Now let $B_S=\psi_*B_T$ where ψ denotes $T\to S$.

Fact: $B_S\geq 0$. This can be reduced to the case when $\dim X=2$.

Lemma 15.6. *Let X be a normal projective variety, D, D' $\mathbb{Q}$-Cartier divisors on X. Let $f\colon X \to Z$ be a D-flipping and D'-flipping contraction. Then the flip for D exists if and only if the flip for D' exists.*

Proof. Let R be the extremal ray corresponding to $X \to Z$. Then $D \cdot R < 0$ and $D' \cdot R < 0$. There exists some $e > 0$ such that $(D - eD') \cdot R = 0$. By the cone and contraction Theorem 11.1,

$$D - eD' \sim_{\mathbb{Q}} f^* L$$

for some $\mathbb{Q}$-Cartier divisor L on Z. Assume the D-flip exists:

$$\begin{tikzcd} D, X \arrow[rr, dashed, "\varphi"] \arrow[dr, "f"'] & & X^+, D^+ \arrow[dl, "f^+"] \\ & Z & \end{tikzcd}$$

where $D^+ = \varphi_* D$. Let $D'^+ = \varphi_* D'$. Then

$$D^+ - eD'^+ \sim_{\mathbb{Q}} (f^+)^* L.$$

Since D^+ is ample over Z, D'^+ is ample over Z. So the above is also the flip for D'. Similarly, if the D'-flip exists, then the D-flip exists. □

Definition 15.7 (pl-flipping contractions)**.** Let (X, B) be a $\mathbb{Q}$-factorial dlt pair, projective. Let $f\colon X \to Z$ be a $K_X + B$ flipping contraction ($-(K_X + B)$ ample over Z). We say f is a *pl-flipping contraction* if there exists a component S of $\lfloor B \rfloor$ such that $-S$ is ample over Z. If the flip exists, we say it is a pl-flip.

Remark 15.8. One of the key insights of Shokurov has been to reduce existence of pl-flips of klt pairs to existence of pl-flips. Hopefully we see this in the next semester.

15.2 Reduction of existence of pl-flips to lower dimension

Assume (X, B) is $\mathbb{Q}$-factorial dlt, projective, and $f\colon X \to Z$ is a pl-flipping contraction. The goal is to show the corresponding pl-flip exists. We sketch an approach of Shokurov (with input from Hacom-McKernan) to reduce the problem to statements in lower dimension.

Step 1. The problem is local over Z, so can assume $Z = \operatorname{Spec} A$ is affine. By Lemma 15.6, enough to show the S-flip exists. By Theorem 14.6, it is enough to show that

$$\mathcal{R}(S/Z) = \bigoplus_{m \geq 0} H^0(X, mS)$$

is a finitely generated A-algebra. Also can assume $\lfloor B \rfloor = S$.

Step 2. Since $f\colon X \to Z$ is a small contraction, can find

$$D \sim S, \quad S \not\subseteq \operatorname{Supp} D.$$

To see this use the fact that Z is affine so $\mathcal{O}_Z(f_* S)$ is generated by global sections. Then $\mathcal{R}(S/Z)$ is a finitely generated A-algebra if and only if $\mathcal{R}(D/Z)$ is a finitely generated A-algebra. Note since $D \sim S$, there exists a rational function $\tau \in k(X)$ such that

$$D + \operatorname{Div}(\tau) = S.$$

Then $\tau \in \mathcal{R}(D/Z)$ of degree one.

Step 3. For each $m \geq 0$, we have an exact sequence

$$0 \longrightarrow \mathcal{O}_X(mD - S) \longrightarrow \mathcal{O}_X(mD) \longrightarrow \mathcal{F}_m \longrightarrow 0$$

for some sheaf $\mathcal{F}_m$ supported on S. In general, $\mathcal{F}_m$ may not be equal to $\mathcal{O}_S(mD|_S)$. We get exact sequence

$$0 \longrightarrow H_0(X, mD - S) \longrightarrow H^0(X, mD) \longrightarrow H^0(S, , \mathcal{F}_m)$$

and

$$0 \longrightarrow H^0(X, mD - S) \longrightarrow H^0(X, mD) \longrightarrow H^0(X, mD)|_S \longrightarrow 0.$$

Taking direct sum gives an exact sequence

$$\begin{tikzcd}
0 \arrow[r] & \bigoplus_{m\geq 0} H^0(X, mD-S) \arrow[r] & \bigoplus_{m\geq 0} H^0(X, mD) \arrow[r] \arrow[d, equal] & \bigoplus_{m\geq 0} H^0(X, mD)|_S \arrow[r] \arrow[d, equal] & 0 \\
& & \mathcal{R}(D/Z) \arrow[r, "\theta"] & \mathcal{R}(D/Z)|_S \arrow[r] & 0
\end{tikzcd}$$

for a certain algebra $\mathcal{R}(D/Z)|_S$ on S which is simply the image of $R(D/Z)$.

Step 4. Assume $\alpha \in \operatorname{Ker}\theta$ of degree m. Then $\operatorname{Div}(\alpha) + mD - S \geq 0$. So

$$\begin{aligned} Div(\alpha) + (m-1)D + D - S &= \operatorname{Div}(\alpha) + (m-1)D + \operatorname{Div}(\frac{1}{\tau}) \\ &= \operatorname{Div}(\frac{\alpha}{\tau}) + (m-1)D \geq 0. \end{aligned}$$

Hence $\beta \coloneqq \frac{\alpha}{\tau} \in \mathcal{R}(D/Z)$ of degree $m-1$. Then $\alpha = \beta\tau$. Thus $\alpha \in \langle\tau\rangle$, which is an ideal in $\mathcal{R}(D/Z)$ generated by τ. Therefore,

$$\operatorname{Ker}\theta = \langle\tau\rangle,$$

as $\tau \in \operatorname{Ker}\theta$ because τ vanishes on S, by our choice of τ.

Step 5. We show that $\mathcal{R}(D/Z)$ is a finitely generated A-algebra if and only if $\mathcal{R}(D/Z)|_S$ is a finitely generated A-algebra.

($\Longrightarrow$) Obvious.

($\Longleftarrow$) Say $\mathcal{R}(D)|_S$ is generated by $\theta(\alpha_1), \ldots, \theta(\alpha_r)$, where α_i are homogeneous elements of $\mathcal{R}(D/Z)$. Pick $\alpha \in \mathcal{R}(D/Z)$ homogeneous of degree m. Can write $\alpha = \alpha' + \alpha''$ where

$$\begin{cases} \alpha' \text{ generated by } \alpha_1, \ldots, \alpha_r, \\ \alpha'' \in \operatorname{Ker}\theta. \end{cases}$$

Then $\alpha'' = \beta\tau$ for some β of degree $m-1$. Repeating the argument with β in place of α shows that α belongs to an algebra generated by $\alpha_1, \ldots, \alpha_r, \tau$. Therefore, $\mathcal{R}(D/Z)$ is generated by $\alpha_1, \ldots, \alpha_r, \tau$.

Step 6. By the cone and contraction Theorem 11.1,

$$ID \sim J(K_X + B)$$

over Z with ID and $J(K_X + B)$ Cartier for some $I, J \in \mathbb{N}$. Shrinking Z, can assume $ID \sim J(K_X + B)$. Then, $\mathcal{R}(D/Z)|_S$ is a finitely generated A-algebra if and only if $\mathcal{R}(ID/Z)|_S$ is a finitely generated A-algebra if and only if $\mathcal{R}(J(K_X + B)/Z)|_S$ is a finitely generated A-algebra. By adjunction we have

$$(K_X + B)|_S = K_S + B_S$$

for some $B_S \geq 0$. Since (X, B) is plt, (S, B_S) is klt. Also can assume $S \not\subseteq \operatorname{Supp}(K_X + B)$. Now

$$\mathcal{R}(J(K_X + B)/Z)|_S \subseteq \mathcal{R}(J(K_S + B_S)/Z).$$

If equality holds, then we use induction as we can assume $\mathcal{R}(J(K_X + B_S)/Z)$ is finitely generated. But in general equality may not hold.

Step 7. The ideas is to work on a higher resolution. Assume $\varphi\colon W \to X$ is a log resolution. Let $\Delta_W = B_W^{\geq 0}$. Then

$$K_W + \Delta_W = \varphi^*(K_X + B) + E$$

where $E \geq 0$ is φ-exceptional, Δ_W, E have no common components. Then $\mathcal{R}(J(K_W + \Delta_W)/Z) = \mathcal{R}(J(K_X + B)/Z)$. Let T be the birational transform of S, and ψ the morphism $T \to S$. Then

$$\psi_* \mathcal{R}(J(K_W + \Delta_W)/Z)|_T = \mathcal{R}(J(K_X + B)/Z)|_S.$$

So enough to show $\mathcal{R}(J(K_W + \Delta_W)/Z)|_T$ is finitely generated.

Step 8. Define $K_T + \Delta_T = (K_W + \Delta_W)|_T$ by adjunction. Replacing W by taking some blowups we can assume (T, Δ_T) is "terminal", i.e., for any resolution $\lambda\colon U \to T$, writing $K_U + B_U = \lambda^*(K_T + B_T)$, coefficients of exceptional components of B_U are negative.

Moreover, making some slight changes to B, Δ_W, can assume $\Delta_W = \Delta'_W + H_W$ where

$$\begin{cases} \Delta'_W \geq 0, \quad \lfloor \Delta'_W \rfloor = T, \\ H_W \geq 0 \text{ is ample.} \end{cases}$$

We are thus in the situation to apply the followings, perhaps after replacing J with a multiple.

Theorem 15.9 (Extension theorem)**.** *Assuming existence of minimal model, etc. in lower dimension (to be discussed later in next semester),*

$$\mathcal{R}(J(K_W + \Delta_W)/Z)|_T = \mathcal{R}(J(K_T + \Lambda_T)/Z)$$

for some boundary $\Lambda_T \leq \Delta_T$.

Actually, Λ_T is determined as follows. For each $m > 0$, let

$$F_{m!} = \operatorname{Fix}|m!(K_W + \Delta_W)|_T.$$

Note that $F_{m!}$ is the largest divisor such that $F_{m!} \leq M|_T$, for any $0 \leq M \sim m!(K_W + \Delta_W)$ and $T \not\subseteq \operatorname{Supp} M$. Put $F = \lim_{m\to\infty} F_{m!}$. Then $\Lambda_T = \Delta_T - (\Delta_T \wedge F)$ where $\Delta_T \wedge F$ is the largest divisor such that

$$\begin{cases} \Delta_T \wedge F \leq \Delta_T, \\ \Delta_T \wedge F \leq F. \end{cases}$$

Reference for the extension theorem: [Hacom-McKernan, Existence of minimal models for varieties of log general type II].