MOISHEZON MANIFOLDS WITH NO NEF AND BIG CLASSES

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Abstract. We show that a compact complex manifold X has no non-trivial nef $(1,1)$-classes if there is a non-isomorphic bimeromorphic map $f\colon X \dashrightarrow Y$ isomorphic in codimension 1 to a compact Kähler manifold Y with $h^{1,1}=1$. In particular, there exist infinitely many isomorphic classes of smooth compact Moishezon threefolds with no nef and big $(1,1)$-classes. This contradicts a recent paper (Strongly Jordan property and free actions of non-abelian free groups, Proc. Edinb. Math. Soc., (2022): 1–11).

Contents

1. Introduction

Let X be a compact complex manifold with a fixed positive Hermitian form ω. Let α be a closed $(1,1)$-form. We use $[\alpha]$ to represent its class in the Bott-Chern $H^{1,1}_{BC}(X)$. Recall the following positivity notions (independent of the choice of ω).

- $[\alpha]$ is *Kähler* if it contains a Kähler form, i.e., if there is a smooth function φ such that $\alpha + \sqrt{-1}\partial\overline{\partial}\varphi \geq \epsilon\omega$ on X for some $\epsilon > 0$.
- $[\alpha]$ is *nef* if for every $\epsilon > 0$ there is a smooth function φ_ϵ such that $\alpha + \sqrt{-1}\partial\overline{\partial}\varphi_\epsilon \geq -\epsilon\omega$ on X.
- $[\alpha]$ is *big* if it contains a Kähler current, i.e., if there exists a quasi-plurisubharmonic function (quasi-psh) $\varphi\colon X \longrightarrow \mathbb{R}\cup\{-\infty\}$ such that $\alpha + \sqrt{-1}\partial\overline{\partial}\varphi \geq \epsilon\omega$ holds weakly as currents on X, for some $\epsilon > 0$.

We say X is in *Fujiki's class $\mathcal{C}$* (resp. *Moishezon*) if it is the meromorphic image of a compact Kähler manifold (resp. projective variety), or equivalently it is bimeromorphic to a compact Kähler manifold (resp. projective variety). It is also equivalent to X admitting a big $(1,1)$-class (resp. big Cartier divisor). We refer to [4, Definition 1.1 and Lemma 1.1], [21, Chapter IV, Theorem 5] and [2, Theorem 0.7] for equivalent definitions and some properties of Fujiki's class $\mathcal{C}$.

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Throughout this article, we work in the Fujiki's class $\mathcal{C}$ where $\partial\overline{\partial}$-lemma holds. So we are freely to use the equivalent Bott-Chern and de Rham cohomologies.

We start with the following main observation.

Theorem 1.1. *Let $f\colon X \dashrightarrow Y$ be a bimeromorphic map of compact complex manifolds which is isomorphic in codimension* 1*. Suppose X is Kähler with $h^{1,1}(X,\mathbb{R}) = 1$ and f is non-isomorphic. Then any nef $(1,1)$-class on Y is trivial. In particular, Y is a non-Kähler manifold in Fujiki's class $\mathcal{C}$ with no nef and big $(1,1)$-classes.*

One way to construct $f\colon X \dashrightarrow Y$ in Theorem 1.1 is by considering an elementary transformation or a (non-projective) flop.

Example 1.2. Let $X \subset \mathbb{P}^4$ be a generic smooth quintic threefold. By a classical result of Clemens and Katz (cf. [1, 13]), X contains a smooth rational curve C_d of degree d with normal bundle $\mathcal{N}_{C_d/X} \cong \mathcal{O}_{C_d}(-1)^{\oplus 2}$. This result was later generalised to a complete intersection of degree $(2,4)$ in $\mathbb{P}^5$ by Oguiso (cf. [18, Theorem 2]). Let $p\colon Z_d \to X$ be the blowup along C_d. Then the exceptional divisor $E \cong C_d \times C_d' \cong \mathbb{P}^1 \times \mathbb{P}^1$. By the contraction theorem of Nakano-Fujiki (cf. [3]), there is a bimeromorphic morphism $q\colon Z_d \to Y_d$ to a smooth compact complex manifold Y_d which contracts E to C_d' along C_d. Then we can construct $f \coloneqq q \circ p^{-1}\colon X \dashrightarrow Y_d$ which is isomorphic in codimension 1. By the Lefschetz hyperplane theorem, we see that $h^2(X,\mathbb{R}) = 1$ and hence $h^{1,1}(X,\mathbb{R}) = 1$. Applying Theorem 1.1, we obtain infinitely many isomorphic classes of smooth Calabi-Yau Moishezon threefolds $\{Y_d\}_{d>0}$ satisfying the following theorem.

Theorem 1.3. *There exist infinitely many isomorphic classes of smooth compact Moishezon threefolds with no nef and big $(1,1)$-classes.*

Nakamura (cf. [16, (3.3) Remark]) provides another example for the above theorem.

Example 1.4. There is a bimeromorphic map $f\colon \mathbb{P}^3 \dashrightarrow X$ to a smooth Moishezon threefold X of $h^{1,1}(X,\mathbb{R}) = 1$ with no nef and big $(1,1)$-class. The map f is constructed by first blowing up a non-singular curve of bidegree $(3,k)$ with $k \geq 7$ in a smooth quadric surface $S \cong \mathbb{P}^1 \times \mathbb{P}^1$ and then contracting the proper transform of S. Then $H^{1,1}(X,\mathbb{R})$ is generated by a big divisor L with $L^3 < 0$. So X admits no nef and big class. Note that, in this case, f is not isomorphic in codimension 1.

The aim of the present note is to confute a key theorem in the recent paper [14] as explained in the following remark.

Remark 1.5. In [14, Theorem 4.2(1)], the author asserts that a compact complex manifold X in Fujiki's class $\mathcal{C}$ always admits a nef and big class. However, as we just discussed, Examples 1.2 and 1.4 or Theorem 1.3 confute this claim. Note that [14, Theorem 4.2(1)] plays a crucial role in the proof of [14, Corollary 4.3] that $\mathrm{Aut}_\tau(X)/\mathrm{Aut}_0(X)$ is finite where $\mathrm{Aut}_\tau(X)$ is the group of automorphisms (pullback) acting trivially on $H^2(X,\mathbb{R})$ and $\mathrm{Aut}_0(X)$ is the neutral component. So the proof there does not work. Nevertheless, the statement [14, Corollary 4.3] still holds and was previously proved by showing the existence of equivariant Kähler model; see [12, Theorem 1.1, Corollary 1.3].

It is known that a smooth compact surface in Fujiki's class $\mathcal{C}$ is Kähler and hence a smooth Moishezon surface is projective. So Theorem 1.3 is optimal in terms of minimal dimension and it is easy to construct examples, like those in Theorem 1.3, of arbitrary higher dimensions by further taking the product with a smooth projective variety of suitable dimension. In the singular surface case, we summarize several examples constructed by Schröer (cf. [20]) and Mondal (cf. [15]) in the following remark.

Remark 1.6. The examples in [20] are constructed in a similar way by different elementary transformations of $\mathbb{P}^1 \times C$ where the genus $g(C) > 0$. However, they behave quite differently on Cartier divisors. The example in [15, § 2] is a supplement to (1) on the rational case. It seems that we do not know any rational example for point (3).

(1) (cf. [20, § 3]) There is a non-projective normal compact Moishezon surface S such that the Picard number of S is 0. In particular, S admits no non-trivial nef Cartier divisor.
(2) (cf. [15, § 2]) There is a non-projective normal compact Moishezon **rational** surface S such that the Picard number of S is 0. The surface is Y_2' in [15, § 2]. We give some explanation on the Picard number. Note that the Weil-Picard number of S is 1 (cf. [17, Definition 2.7 and Lemma 2.10]). Since S is not projective (cf. [15, Theorem 4.1 and Example 3.19]), the Picard number of S has to be 0 (cf. [17, Definition 2.11–Remark 2.13, Remark on the top of page 303]).
(3) (cf. [20, § 4]) There is a non-projective normal compact Moishezon surface Z which allows a non-projective birational morphism $Z \to S$ to a projective surface S. In particular, Z admits a nef and big Cartier divisor which is the pullback of an ample Cartier divisor on S.

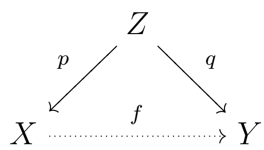
2. Proof of Theorem 1.1

We first reprove [6, Theorem 4.5] by the following Proposition 2.1. The first version of this proposition was formulated in [5, Corollary 3.3] where Fujiki works in the smooth setting and $f_*[\alpha]$ is assumed to be semi-positive. Later, it was generalized by Huybrechts (cf. [11, Proposition 2.1]) to the situation when canonical bundles K_X and K_Y are nef and $[\alpha]$ and $f_*[\alpha]$ are only assumed to have positive intersections with all rational curves.

When dealing with the singular setting in the below Proposition 2.1, we refer to [10] for the basic definitions involved. Note that for a normal compact complex space X with rational singularities, $H^{1,1}(X,\mathbb{R})$ embeds into $H^2(X,\mathbb{R})$ naturally, and the intersection product on $H^{1,1}(X, \mathbb{R})$ can be defined via the cup-product for $H^2(X, \mathbb{R})$ (cf. [10, Remark 3.7]). Of course, for the purpose of this note, one can focus on the smooth setting for simplicity.

Proposition 2.1 (cf. [6, Theorem 4.5] and Remark 2.3)**.** *Let $f\colon X \dashrightarrow Y$ be a bimeromorphic map of normal compact complex spaces with rational singularities. Suppose f does not contract divisors and there exists a Kähler class $[\alpha] \in H^{1,1}(X, \mathbb{R})$ such that $f_*[\alpha]$ is nef. Then f^{-1} is holomorphic.*

Proof. Consider the log resolution of the indeterminacy of f:



where $p\colon Z \to X$ and $q\colon Z \to Y$ are the two projections. By Chow's lemma (cf. [9, Corollary 2 and Definition 4.1]), we may assume p is a projective morphism obtained by a finite sequence of blowups along smooth centres. Denote by $\bigcup_{i=1}^n E_i$ the full union of exceptional prime divisors of p. One can find

$$E = \sum_{i=1}^{n} \delta_i E_i$$

with suitable $\delta_1, \ldots, \delta_n > 0$ such that $-E$ is p-ample (cf. [2, Proof of Lemma 3.5]). Here, if $n = 0$, then p is isomorphic and $-E = 0$ is automatically p-ample.

Note that $p^*[\alpha]$ is represented by a smooth semi-positive form and q-exceptional divisors are also p-exceptional divisors since f does not contract divisors by the assumption. Applying [6, Lemma 4.4] (cf. [5, Lemma 2.4]) to $p^*[\alpha]$, we have

$$q^*q_*p^*[\alpha] - p^*[\alpha] = \sum_{i=1}^{n} a_i [E_i]$$

with $a_i \geq 0$.

Claim 2.2. *We claim that $q^*q_*p^*[\alpha] - p^*[\alpha] = 0$.*

Proof. Suppose the contrary that $a_1 > 0$ without loss of generality. Note that $q^*q_*p^*[\alpha]$ is nef and $p^*[\alpha]$ is p-trivial. Then the divisor

$$D := \sum_{i=1}^{n} a_i E_i - \epsilon E = \sum_{i=1}^{n} (a_i - \epsilon \delta_i) E_i$$

is p-ample and $-D$ is not effective whenever $0 < \epsilon < a_1/\delta_1$. We can further find rational coefficients b_i sufficiently closed to $a_i - \epsilon\delta_i$ such that

$$D' := \sum_{i=1}^{n} b_i E_i$$

is still p-ample and $-D'$ is not effective. Note that mD' is then a Cartier divisor for a suitable integer m and $p_*(-mD') = 0$. By the negativity lemma for Cartier divisors (cf. [22, Lemma 1.3]), $-mD'$ is effective, a contradiction. So the claim is proved. □

Applying Chow's lemma again, there is a bimeromorphic morphism $\sigma\colon W \to Z$ such that $q \circ \sigma$ is a projective morphism. Note that $(q \circ \sigma)_*(p \circ \sigma)^*[\alpha] = q_*p^*[\alpha]$. So we may replace Z by W and assume q is already projective (without requiring p being projective). Let F be any fibre of q which is projective. Let C be any curve in F. By the projection formula and Claim 2.2,

$$\int \alpha \wedge \langle p_*C\rangle = \int p^*\alpha \wedge \langle C\rangle = \int q^*q_*p^*\alpha \wedge \langle C\rangle = \int q_*p^*\alpha \wedge \langle q_*C\rangle = 0$$

where $\langle - \rangle$ represents the integration current. Since $[\alpha]$ is Kähler, $p(C)$ is a point and hence $p(F)$ is a point. By the rigidity lemma (cf. [6, Lemma 4.1]) which is essentially due to the Riemann extension theorem (cf. [8, Page 144]), $f^{-1}\colon Y \to X$ is a holomorphic map. □

Remark 2.3. Claim 2.2 was treated in the proof of [6, Theorem 4.5, Equation (4.4)]. However, the proof there seems incomplete after Equation (4.2) where the author claims "the singular locus of the nef class is empty". This is also mentioned after [6, Definition 4.3] where the author seems to have misinterpreted a result of Boucksom. Note that a nef class has empty singular locus if and only if it is semi-positive. However, there are situations where non-semi-positive nef classes exist. Nevertheless, we can overcome this gap by applying the negativity lemma as in the proof of Claim 2.2.

Proof of Theorem 1.1. Note that $h^{1,1}(Y,\mathbb{R}) = h^{1,1}(X,\mathbb{R}) = 1$ because f is isomorphic in codimension 1 (cf. [19, Corollary 1.5]). Let $[\alpha]$ be a Kähler class on X. Then $H^{1,1}(Y,\mathbb{R})$ is generated by the big class $f_*[\alpha]$. Let $[\gamma] \in H^{1,1}(Y,\mathbb{R})$ be a nef class. Then $[\gamma] = tf_*[\alpha]$ for some $t \geq 0$ (cf. [5, Lemma 2.1]). So it suffices to show that $f_*[\alpha]$ is not nef.

Suppose the contrary that $f_*[\alpha] \in H^{1,1}(Y,\mathbb{R})$ is nef. By Proposition 2.1, f^{-1} is holomorphic. By the purity (cf. [7, Satz 4]) and since f^{-1} is isomorphic in codimension 1, the exceptional locus of f^{-1} is empty. In particular, f is isomorphic, a contradiction. □

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